Regular Maps on a Given Surface: A Survey

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Summary. Regular maps are cellular decompositions of closed surfaces with the highest 'level of symmetry', meaning that the automorphism group of the map acts regularly on flags. We survey the state-of-the-art of the problem of classification of regular maps on a given surface and outline directions of future research in this area.

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1 What Is a Map?

This question was the title of a paper by W. T. Tutte [Tut73] that appeared after more than one hundred years of fruitful research into maps on surfaces. In fact, origins of what is now known as theory of maps go back to ancient Greece. Mathematics of that period was dominated by the beauty of geometry, including fascination about polyhedra with certain regularity properties. The most prominent examples are the ubiquitous five Platonic solids. The reader may think of these as of our first examples of maps. Each of the five polyhedra has vertices, edges, and faces, and may thus be viewed as a "drawing" of a graph on the sphere. In the case of the dodecahedron, for instance, it would be a spherical "drawing" containing 20 vertices, 30 edges, and 12 pentagonal faces.

Most spatial models of the dodecahedron would have all edges of the same length and all faces bounded by congruent, regular pentagons. These geometric features are, however, not of concern in the theory of maps. From now on we will be interested only in the way how vertices, edges, and faces interact, regardless of the geometric shape of edges, faces, and their boundaries. In

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particular, we will not require that edges be straight-line segments, faces be flat, and their boundaries be regular polygons. Only the cell structure of the resulting object will matter.

In the course of our exposition we will assume that the reader is, at least at an elementary level, familiar with fundamentals of topology and with the concept of a compact, orientable or nonorientable, 2-dimensional surface, or, shortly, a *surface*. Also, we will assume familiarity with basic notions of graph theory and, later, group theory as well.

Thus, what is a map? Intuitively, it is "a drawing of a graph on a surface". To make this precise, let us regard graphs as topological 1-dimensional complexes. Then, an *embedding* of a graph Γ on a surface $\mathcal{S}$ is a continuous one-to-one mapping $f : \Gamma \to \mathcal{S}$. One usually identifies the image $f(\Gamma)$ on the surface $\mathcal{S}$ with the graph Γ itself. Connected components of $\mathcal{S} \setminus f(\Gamma)$ are called *faces*. The embedding f is *cellular* if every face is homeomorphic to an open disc. Note that cellularity implies connectivity of the embedded graph. Also, in the cellular case, the boundary of each face is formed by a closed walk in the embedded graph. Finally, any cellular embedding of a graph will be called a *map*.

How can one actually describe maps? In other words, what do we have to specify in order to uniquely determine an embedding of a connected graph? The answer is simple if we restrict ourselves to orientable surfaces. If a graph Γ is cellularly embedded on an orientable surface, a choice of a preferred orientation of the surface induces, at each vertex, a cyclic permutation of edges leaving that vertex. It is customary to represent an edge leaving a vertex by an arrow on the edge pointing out of the vertex – or, to think of an edge with direction, although our graphs are undirected. This way, a map on an oriented surface induces a permutation of edges with directions, such that the (entries in) cycles of the permutation exactly correspond to (edges directed out of) vertices. Such a permutation is called a *rotation*. Conversely, each rotation of a connected graph gives rise to a unique map on an oriented surface. To see this, imagine that each edge is a centre of a narrow band. A cycle of a rotation then determines the cyclic order of the bands leaving a disc neighbourhood of a vertex. As the result we obtain a *band complex* in which the graph is embedded. It just remains to fill the "holes" by cells to obtain the embedding. The situation is more complicated in the nonorientable case because of lack of global orientation. In addition to a rotation, one would have to specify the way the "local" orientations given by the cycles of the rotation interact. In terms of the band complex, this would tell us which of the bands have to be "twisted". Rather than going into further details that can be found in the monograph by J. L. Gross and T. W. Tucker [GrT87], we pass on to a different approach.

The description using rotation focuses on the graph in the first place; the embedding is then constructed with the help of a band complex. A map, however, can also be viewed as a cellular decomposition of a surface. This suggests looking for a description that would capture cellularity right at the outset. Consider a map on a surface $\mathcal{S}$, formed by an embedded graph Γ, and suppose that Γ is "drawn" on $\mathcal{S}$ in thick lines. Pick a point in the interior of each face and call it the *centre* of the face. In each face, join the centre by dashed and thin line segments with every vertex and the midpoint of every edge, respectively, on the boundary of the face. The resulting refined structure is the *barycentric subdivision* of the map. Cells of the subdivision are topological triangles called *flags*. Each flag has a thick, a thin, and a dashed side (as in Fig. 1 of section 3, ignoring labels). To describe the map in terms of flags and their interaction, let $\mathcal{F}$ be the set of flags of the map. We introduce three permutations X, Y, and Z on $\mathcal{F}$ as follows. For each flag $b \in \mathcal{F}$ and for each $W \in \{X, Y, Z\}$, the flag $W(b)$ is the unique flag different from b that shares with b the thin, the thick, or the dashed side, depending on whether W is equal to X, Y, or Z. Obviously, X, Y, Z are involutions with no fixed points, and it is easy to see that X commutes with Y. Moreover, cellularity implies that the group $\langle X, Y, Z\rangle$ is a transitive permutation group on $\mathcal{F}$. Conversely, any such permutation group gives rise to a map; we will discuss the construction in detail in an important special case in section 3. In this place we just add two remarks. First, the supporting surface of the map is orientable if and only if the subgroup generated by the two products YZ and ZX has index two in $\langle X, Y, Z\rangle$. Second, the above discussion also applies to infinite maps which we will encounter in a few places in section 2.

Now, after all, what is a map? We have seen a topological definition and a combinatorial description. The answer suggested by Tutte's paper [Tut73] is purely algebraic and follows the third way we have just outlined: *A map is a transitive permutation group generated by three fixed-point-free involutions, two of which commute.*

2 What Is a Regular Map?

If an article with this title had been written, perhaps it, too, would have found its answer in permutation groups. This will transpire after we explain the basics. Before we begin, we would like to note that the concept of regularity has a number of meanings in mathematical disciplines. Here, regularity will mean "highest level of symmetry".

How would one define a "symmetry" of a map? Taking, say, the dodecahedron again, there is a large number of "symmetries" – rotations and reflections – that preserve the solid. All of them carry vertices, edges, and faces to the corresponding objects; in particular, they carry flags onto flags. This tells us what we should be looking for in general. Let M be a map with flag set $\mathcal{F}$. We define an *automorphism* of M to be a permutation of $\mathcal{F}$ that maps pairs of flags sharing a thick (dashed, thin) side to pairs sharing the same type of side. It is easy to see that such a permutation behaves as expected: it preserves the cell structure of the map and also induces an automorphism of the embedded graph. The collection of all automorphisms of M forms, under composition of mappings, the *automorphism group* of M, denoted $Aut(M)$. The important observation to make is that the group $Aut(M)$ acts *freely* on $\mathcal{F}$, that is, for any two flags $b, b' \in \mathcal{F}$ there exists at most one $\alpha \in Aut(M)$ such that $\alpha(b) = b'$.

What is now the largest "level of symmetry" a map can have? The most one can expect is that for any ordered pair of flags there is *exactly one* automorphism that maps the first flag onto the second. Such actions of groups are known as regular actions. We therefore define a map M to be *regular* if the group $Aut(M)$ acts regularly on the flag set of M. Maps with the theoretically largest "level of symmetry" are thus the regular maps.

Another, and equivalent, definition of regularity of maps makes no prior reference to automorphisms at all. Let M be a map and let $\langle X, Y, Z \rangle$ is the corresponding transitive permutation group representing M. Then, M is regular if the group $\langle X, Y, Z \rangle$ acts freely (and hence regularly) on the flag set of M. Equivalence of the two definitions follows from known facts about general group actions and we refer to R. P. Bryant and D. Singerman [BrS85] for details. This gives an immediate one-to-one correspondence between regular maps and groups generated by three involutions, two of which commute. It also opens up numerous connections to other branches of mathematics such as hyperbolic geometry, theory of Riemann surfaces, and Galois theory, as we shall see in the next section.

The two ways of introducing regular maps arrive at the goal from two opposite directions. In the first scenario, the automorphism group of a map acts freely on the set of flags. The way to make the automorphism group the *largest* possible is to require that it act transitively on the set of flags. On the other hand, the map can be identified with a certain transitive permutation group on a set, the set of flags of the map. The way to make such a group the *smallest* possible is to stipulate that it act freely on the flag set. Both approaches meet in the concept of a regular map.

If the supporting surface of a map is orientable, one may introduce a weaker concept of regularity by focusing on orientation preserving automorphisms only. In such a case we say that the map is *orientably regular* if its group of orientation preserving automorphisms acts regularly on mutually incident vertex-edge pairs, or, equivalently, on edges with a preassigned direction. A map that is orientably regular but not regular is called *chiral*.

The algebraic viewpoint of regular maps evokes the feeling that one can actually forget about surfaces and topology. This is, to some extent, the case. Such a standpoint, however, would not be productive in general, since it would cut off a considerable supply of combinatorial and topological ideas that have contributed to the theory of regular maps in the past.

Perhaps the most famous examples of regular maps are the five Platonic polyhedra that permeate our exposition. It is the wealth of non-spherical regular maps, however, that give this topic fascinating dimensions. Such maps were considered by medieval astronomers in their attempts to explain the planetary system. Prominent examples are the stellated polyhedra that appeared in the work of J. Kepler [Kep19] as early as in 1619. More than two centuries later, maps and regular maps resurfaced in two independent and, at the beginning, unrelated streams of research. The first was driven mainly by the appearance of the four colour problem and the map colouring problem of P. J. Heawood [Hea90]. In this connection, L. Heffter [Hef98] discovered orientably regular embeddings of complete graphs of prime order. Approximately at the same time, certain three-valent regular maps on a surface of genus three were studied by F. Klein [Kle79] and W. Dyck [Dyc80] in a completely different connection – constructions of multiply periodic (or, automorphic) complex functions. As a way of a geometric representation of groups, regular maps also appeared in the monograph by W. Burnside [Bur11]. The term regular map, however, was first introduced as late as in 1927 by H. R. Brahana in [Bra27], which appears to be the first systematic treatment of the topic.

Development in the classification of regular maps on a given surface in the 20th century will be overviewed in section 4. Here we only note that foundations of modern theory of orientably regular and regular maps have been laid by G. A. Jones and D. Singerman [JoS78], and by R. P. Bryant and D. Singerman in [BrS85]. The importance of the two papers, however, and even more so of the two successive articles by G. A. Jones and D. Singerman [JoS96] and by G. A. Jones [Jon97], also lies in pointing out the fascinating connections between the theory of maps, theory of groups, geometry of surfaces, Riemann surfaces, and Galois theory, which we will briefly summarise in section 3.

Classification of regular, orientably regular, and chiral maps on a given surface thus appears to be an important problem. Besides natural significance in the theory of maps, progress towards a solution of the problem would advance knowledge and find applications in the disciplines mentioned above. The aim of this paper is to survey results that have been achieved in this direction. Following a more detailed presentation of the algebraic background, in section 4 we focus on regular maps on surfaces of relatively small genus. In section 5 we give a brief account on the classification of regular maps on surfaces of Euler characteristic $\chi = -p$ where p is a prime. We also outline the newest development regarding regular maps on surfaces with $\chi = -2p$, $-3p$, and p^2, including a discussion on maps of Zassenhaus type. In the final section 6 we mention possible generalisations to regular hypermaps.

Let us note for completeness that the problem of classification of regular and orientably regular maps has been approached from other angles as well, such as classification by the automorphism groups, by the (fixed) underlying graphs, and by families of graphs. These are out of the scope of our article and would, in fact, deserve a separate survey paper. We therefore conclude with a short selection of influential results in these areas. An enumeration of orientably regular maps with automorphism groups isomorphic to 2-dimensional projective special linear groups can be extracted from results of S. H. Sah [Sah69]. An abstract characterization of graphs underlying regular and orientably regular maps was given by A. Gardiner, R. Nedela, M. Škoviera and the author in [GNS99]. The classification of orientably regular embeddings of complete graphs was initiated by N. L. Biggs [Big71] and completed by L. A. James and G. A. Jones [JaJ85]. Examples of latest progress in classification of orientably regular embeddings of complete bipartite graphs, complete multipartite graphs, and of cubes are due to G. A. Jones, R. Nedela and M. Škoviera [JNS04], J. H. Kwak and Y. S. Kwon [KwK05], and S. F. Du, J. H. Kwak and R. Nedela [DKN04, DKN05]. The phenomenon of chirality of maps is studied in great detail by A. Breda d'Azevedo, G. A. Jones, R. Nedela and M. Škoviera [BJN05]. The interested reader will find further information in references included in the papers listed.

3 Regular Maps, Groups, and Surfaces

We begin with giving details of the construction of regular maps from groups. Let G be a group generated by three involutions, two of which commute. In order to avoid entanglement into subtleties related to infinite degrees of vertices or faces of infinite length, assume that the product of any two generators of our group has a finite order. Then, G has a presentation of the form

$$G = \langle x, y, z |\ x^2 = y^2 = z^2 = (yz)^k = (zx)^m = (xy)^n = \ldots = 1\rangle \qquad (1)$$

where dots indicate a possible presence of other relations. (As usual, in any such presentation we will assume that the exponents are *true* orders of the elements.) The regular map $M = M(G; x, y, z)$ that corresponds to (1) is constructed as follows. Consider, for each $g \in G$, a topological triangle labelled g, and label its sides with generators of G as in Fig. 1. The collection of all such triangles forms the set of *flags* of the map to be constructed. To simplify the matter, we will identify flags with their group labels. For each $g \in G$ and each $w \in \{x, y, z\}$, $w \neq 1$, we now identify the sides labelled w in the flags g and gw in such a way that the corresponding points where the thick, thin, or dashed sides meet are identified as well. This way we obtain a connected surface without boundary. The cellular decomposition of the surface induced by the union Γ of all thick segments forms our regular map $M = M(G; x, y, z)$. The 1-dimensional cell complex Γ is the *underlying graph* of the map. The identification of flags with elements of G was part of our construction. Other objects such as edges, vertices, and faces of the map M can be similarly identified with the left cosets of the subgroups $\langle x, y\rangle$, $\langle y, z\rangle$, and $\langle z, x\rangle$ in the group G, respectively, and their their mutual incidence is determined by non-empty intersection.

The two natural actions – by left and right multiplication – of the group G on the flag set of M (that is, on G itself) have important map-theoretical interpretations. For regular maps, right multiplication of flags $g \in G$ by the generators x, y, and z gives precisely the permutations X, Y and Z introduced in section 1. For the left multiplication, note that if two flags $h, h' \in G$ are related by some type of reflection, then for any $g \in G$ the flags gh and gh' are related by the same type of reflection. Left multiplication therefore preserves

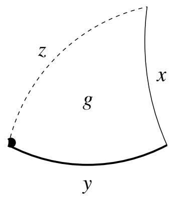

Fig. 1. A topological triangle representing a flag

the cell structure of M and induces an automorphism of M. This way the automorphism group $Aut(M)$ and its action on the map $M = M(G; x, y, z)$ may be identified with the group G and its action on itself by left multiplication. Consequently, regular maps of face length m and vertex valence k can be identified with presentations of finite 3-generator groups as in (1) where k and m represent *true orders* of the elements yz and zx. Briefly, in such a case we speak about regular maps of *type* $\{m, k\}$; listing the face length first is part of the traditional notation known as Schläfli symbol.

Features such as isomorphism and duality of regular maps can be conveniently explained with the help of our approach. Two regular maps $M_i = M(G_i; x_i, y_i, z_i)$, $i = 1, 2$, are *isomorphic* if there is a group isomorphism $\varphi : G_1 \to G_2$ such that $\varphi(x_1) = x_2$, $\varphi(y_1) = y_2$, and $\varphi(z_1) = z_2$. It is easy to check that this condition is equivalent to φ inducing an incidence preserving bijection between flags of M_1 and M_2. Setting $x' = y$ and $y' = x$ in the presentation (1), the map $M(G; x', y', z)$ formed from flags as in Fig. 1 but with thin side labelled $x' = y$ and thick side labelled $y' = x$ is the *dual map* of $M(G; x, y, z)$. The dual of a map M of type $\{m, k\}$ is usually denoted M^* and has type $\{k, m\}$. An illustration is in Fig. 2 which shows a regular embedding of the Petersen graph on the projective plane in solid lines and its dual regular map with underlying graph K_6 in dashed lines. Note that $(M^*)^* = M$ and that both M and M^* have the *same* (not merely isomorphic) automorphism groups. If M^* is isomorphic to M, then M is called *self-dual.*

In section 2 we noted that the theory of regular maps can be completely reduced to group theory. Indeed, it is obvious that instead of a regular map $M = M(G; x, y, z)$ we can just consider the group G *together* with its presentation (1); these will be referred to as $(k, m, 2)$*-groups.* Classification of regular maps of type $\{m, k\}$ up to isomorphism and duality is therefore equivalent to classification of $(k, m, 2)$-groups with $k \geq m$.

Consider now the two particular elements $r = yz$ and $s = zx$ of G. It can be checked that r and s represent a rotation of order k about the vertex and a rotation of order m about the centre of a face, both the vertex and the face centre being incident to the flag labelled 1. The subgroup $G^o = \langle r, s\rangle$ has index at most 2 in G, furnishing a simple orientablilty test: The supporting surface of the map is orientable if and only if the index $[G : G^\circ]$ is equal to 2. In general, if a map of type $\{m, k\}$ on an orientable surface contains two rotations r and s as described above, then the map is said to be *orientably*

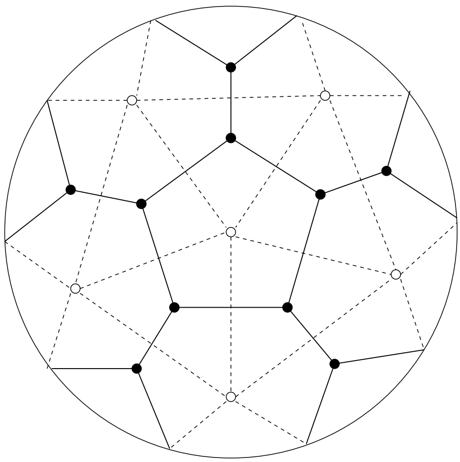

Fig. 2. A regular map and its dual in the projective plane

regular. Thus, every regular map on an orientable surface is orientably regular. Orientably regular maps that are not regular are called *chiral* in the literature. A prominent example of a chiral map is the (essentially, unique) triangular embedding of K_7 on the torus.

We assume familiarity with the concept of the Euler characteristic of a connected compact surface, which is equal to $2-2g$ or $2-h$ depending on whether the surface is orientable, of genus g, or nonorientable, of genus h. Any regular map M on a nonorientable surface of Euler characteristic χ has a natural double cover, the regular map $\tilde{M}$ on the corresponding orientable surface of Euler characteristic 2χ. Reversing the process, M arises from $\tilde{M}$ as a quotient by an antipodal reflection. At the algebraic level, an antipodal reflection of a map is simply a fixed-point-free orientation reversing automorphism commuting with all the orientation preserving automorphisms (and hence lying in the centre of the group, cf. [BeG89]). Such reflections are by no means unique in general. Different antipodal reflections applied to a regular map in the double cover may even yield non-isomorphic regular maps in the quotient surface, as was pointed out by S. Wilson [Wi78a].

The formalism introduced above enables us also to outline the links between the theory of regular maps, group theory, hyperbolic geometry, and complex functions. The extra notion we need is the one of the *full* $(k,m,2)$-*triangle group*, which is the group with presentation $\langle x,y,z \mid x^2=y^2=z^2=(yz)^k=(zx)^m=(xy)^2=1\rangle$. Because of absence of other relations, the corresponding regular map can be realised as a tessellation of a simply connected surface. This surface is the sphere, the euclidean plane, or the hyperbolic plane, depending on whether $1/k+1/m$ is greater than, equal to, or smaller than $1/2$. The tessellation is then formed by geometrically congruent regular m-gons, k of which meet at each vertex. The subgroup of all orientation-preserving automorphisms of this tessellation is the $(k, m, 2)$-*triangle group* $\langle r, s| \ r^k = s^m = (rs)^2 = 1\rangle$, which is an index-two subgroup of the full $(k, m, 2)$-triangle group obtained by letting $r = yz$ and $s = zx$.

The connections are now as follows. Except for embeddings of semi-stars in a sphere, automorphism groups of regular maps of a given type $\{m, k\}$ on compact surfaces are precisely the finite $(k, m, 2)$-groups, which are quotients of the full $(k, m, 2)$-triangle group by torsion-free normal subgroups of finite index. Likewise, finite orientably regular maps of type $\{m, k\}$ are in a one-to-one correspondence with normal, torsion-free, finite-index subgroups of the $(k, m, 2)$-triangle groups; if the subgroup is, at the same time, *not* normal in the *full* triangle group, the corresponding orientably regular map is *chiral.* These quotient constructions can be used to endow maps on compact surfaces with complex structure and geometry (spherical, euclidean, or hyperbolic). In particular, maps (not necessarily regular) can be regarded as complex algebraic curves over algebraic number fields. To conclude with a far-fetching connection, the algebraic curves view opens up a possibility to study the absolute Galois group by its action on maps, as suggested in the Grothendieck's programme [Gro84]. We recommend the survey papers by G. A. Jones [Jon97] and by G. A. Jones and D. Singerman [JoS96] for more details.

4 Regular Maps on Surfaces of Small Genus

In this section we survey regular maps on surfaces of orientable genus up to 15 and nonorientable genus up to 30. Until very recently, these were the only values of genera for which a classification of regular maps was known. In what follows we will be giving presentations of $(k, m, 2)$-groups in the form $\langle (x, y, z), \ \ldots\rangle$ where the (x, y, z) part will stand for $x, y, z|x^2 = y^2 = z^2 = (xy)^2$. For description of groups we will use the standard notation. That is, Z_n, D_n, S_n and A_n will denote the cyclic group of order n, the dihedral group of order $2n$, and the symmetric and the alternating groups of degree n, respectively.

Let G be a finite $(k, m, 2)$-group with a presentation of the form (1). The number of vertices, edges, and faces of the regular map $M = M(G; x, y, z)$ is simply obtained by dividing the number of flags, that is, the order $|G|$ of the group, by the orders of the dihedral stabilisers of the respective elements. The map M therefore has $v = |G|/(2k)$ vertices, $e = |G|/4$ edges, and $f = |G|/(2m)$ faces. By the Euler's formula we have $v - e + f = \chi$ where χ is the Euler characteristic of the supporting surface. Substituting for v, e, f gives

$$\chi = (1/k + 1/m - 1/2)|G|/2 \qquad (2)$$

The surface with the largest Euler characteristic, 2, is the sphere. Regular maps on the sphere have been well known and their classification quickly follows from (2). It turns out that there are no chiral maps on the sphere.

Apart from embeddings of semi-stars (semi-edges incident to a single vertex), a spherical regular map arises either from an embedded k-cycle, or its dual (called a k-dipole), or one of the five Platonic polyhedra. We list the corresponding $(k, m, 2)$-groups $Aut(M)$ (which are all extended triangle groups since the sphere is simply connected) for the maps M up to duality, in the form $M/\text{dual}(M)$:

embedded $k-$dipoles/$k-$cycles	$\langle (x,y,z),\ (yz)^k = (zx)^2 = 1\rangle \cong D_k \times Z_2$
tetrahedron (self $-$ dual)	$\langle (x,y,z),\ (yz)^3 = (zx)^3 = 1\rangle \cong S_4$
octahedron/cube	$\langle (x,y,z),\ (yz)^4 = (zx)^3 = 1\rangle \cong S_4 \times Z_2$
icosahedron/dodecahedron	$\langle (x,y,z),\ (yz)^5 = (zx)^3 = 1\rangle \cong A_5 \times Z_2$

The next simplest surface is the projective plane, which ia a nonorientable surface of Euler characteristic 1 (and of nonorientable genus 1). Since the sphere admits a unique antipodal reflection, regular maps on the projective plane are quotients of the spherical regular maps by the reflection. One has to be cautious, however, when looking for the antipodal reflection in terms of central involutions. For example, the groups G of spherical embeddings of k-dipoles always factor as $G^{\circ} \times \langle x\rangle$, but the automorphism induced by the left multiplication by x has fixed points; the same holds for k-cycles. A closer inspection shows that the spherical embeddings of odd cycles, odd dipoles, and the tetrahedron have no antipodal reflection. Using the notation $r = yz$ and $s = zx$, and taking duality into account, the unique antipodal reflection u in the remaining cases is given by $u = zsr^k$ for $2k$-dipoles, $u = zrs^{-1}r^2s$ for the octahedron, and $u = zr^2sr^{-1}sr^{-2}s$ for the icosahedron. Since the antipodal reflection is central, we have $G = \langle x,y,z\rangle \cong \langle r,s\rangle \times \langle u\rangle \cong G^{\circ} \times Z_2$ and the quotients arise by dividing out by $\langle u\rangle$. Antipodal quotients of our spherical maps are therefore the projective-planar embeddings of k-cycles and their duals (bouquets of k circles), K_4 and its dual (denoted by $K_3^{(2)}$, which is K_3 with doubled edges), and K_6 and its dual (formed by the Petersen graph P embedded as in Fig. 2). Presentations of the corresponding groups $Aut(M)$, again in the form $M/\text{dual}(M)$, are as follows (using $r = yz$ and $s = zx$):

embedded bouquets/cycles	$\langle (x,y,z),\ r^{2k} = s^2 = zsr^k = 1\rangle \cong D_{2k}$
embedded $K_3^{(2)}$/embedded K_4	$\langle (x,y,z),\ r^4 = s^3 = zrs^{-1}r^2s = 1\rangle \cong S_4$
embedded K_6/embedded P	$\langle (x,y,z),\ r^5 = s^3 = zr^2sr^{-1}sr^{-2}s = 1\rangle \cong A_5$

Note that in contrast with the spherical case, presentations of the groups of the projective-planar regular maps contain an extra relator coming from the antipodal involution u. Extra relators feature in all presentations of regular maps on non-simply connected surfaces, since they reflect presence of non-contractible curves.

Classification of regular and chiral maps on the torus (an orientable surface of genus 1, with Euler characteristic 0) was initiated by H. Brahana [Bra26] and all details were eventually supplied by H. S. M. Coxeter [Cox48]. Euler's formula tells us that a toroidal regular map of type $\{m, k\}$, $m \le k$, can exist only if either $k = m = 4$, or $k = 6$ and $m = 3$. By the theory outlined in section 3, for the classification it is sufficient to identify all finite-index, torsion-free, normal subgroups of the $(4, 4, 2)$- and $(6, 3, 2)$-triangle groups and of the related full triangle groups. These are groups of (possibly orientation-reversing) euclidean isometries, leaving invariant the corresponding tessellation. Using representations of the four groups either by matrices or by complex numbers it can be shown that all their finite-index torsion-free normal subgroups are generated by powers of two specific commuting translations. As the result, toroidal chiral and regular maps can be parametrised by ordered pairs of integers (b, c) representing powers of the two translations. Up to isomorphism and duality, the automorphism groups of all *chiral* toroidal maps satisfy $bc(b - c) \neq 0$ and have presentations of the form

$$\text{Type } \{4,4\} : \langle r, s |\; r^4 = s^4 = (rs)^2 = (rs^{-1})^b (r^{-1}s)^c = 1\rangle$$
$$\text{Type } \{6,3\} : \langle r, s |\; r^6 = s^3 = (rs)^2 = (rs^{-1}r)^b (s^{-1}r^2)^c = 1\rangle$$

In the case when $bc(b - c) = 0$ we obtain regular toroidal maps. Using $r = yz$ and $s = zx$, their groups have presentations

$$\text{Type } \{4,4\} : \langle (x, y, z), r^4 = s^4 = (rs^{-1})^b (r^{-1}s)^c = 1\rangle$$
$$\text{Type } \{6,3\} : \langle (x, y, z), r^6 = s^3 = (rs^{-1}r)^b (s^{-1}r^2)^c = 1\rangle$$

Unlike spherical maps, toroidal regular maps do not admit antipodal reflections. Consequently, there are no regular maps on the Klein bottle, the nonorientable surface of Euler characteristic 0 that is double-covered by the torus. Another way to see this is to invoke the known fact that the only automorphism of the Klein bottle acting as a rotation about some point must have order two. This completes the classification of regular maps on surfaces of non-negative Euler characteristic. Note that while the number of regular maps on the sphere, on the projective plane, and on the torus is infinite, the infinitude in the first two cases is due to rather trivial maps.

In contrast with this, the number of regular maps on any compact surface $\mathcal{S}$ of negative Euler characteristic is *finite*. Indeed, from (2) it is obvious that if $\chi(\mathcal{S}) < 0$, then $1/k+1/m < 1/2$ and $|G| = -2\chi(\mathcal{S})/(1/2-1/k-1/m)$. Among all pairs (k, m) for which $1/k + 1/m < 1/2$, the reciprocal of the denominator achieves the largest value, 42, precisely when $\{k, m\} = \{3, 7\}$. We thus arrive at the *Hurwitz bound* $|G| \leq -84\chi(\mathcal{S})$ if $\chi(\mathcal{S}) < 0$, giving a cap on the order of the automorphism group of a regular map on a surface with negative Euler characteristic. Similarly, the order of the automorphism group of a chiral map on an orientable surface of genus $g \geq 2$ cannot exceed $84(g - 1)$. The bound is named after H. Hurwitz who first proved its orientable variant for groups of conformal automorphisms acting on Riemann surfaces; see T. W. Tucker [Tuc83].

To classify regular maps on a given surface of negative Euler characteristic, one could in principle use the strategy outlined in section 3: Work out the admissible types $\{m, k\}$ from the Euler's formula (2) and then determine the torsion-free normal subgroups of the corresponding $(k, m, 2)$-triangle groups of index not exceeding the Hurwitz bound. The second step in full generality, however, seems to be far beyond the reach of the currently available methods. If $1/k + 1/m < 1/2$, that is, when the type $\{m, k\}$ is *hyperbolic*, the full $(k, m, 2)$-triangle group is a subgroup of the group of (direct as well as orientation reversing) hyperbolic isometries, leaving invariant a regular tessellation of the hyperbolic plane of type $\{m, k\}$. As opposed to the euclidean case, very little is known about normal subgroups of such hyperbolic triangle groups.

How can one then approach the problem? In the early stages, a number of results were obtained by relation-chasing, that is, trying to determine the extra relations one has to add in the presentation of a triangle group or a full triangle group to obtain a quotient group and a quotient map on a given surface. Combined with other known facts and methods, mostly of group-theoretical nature, a classification for orientable surfaces of Euler characteristic $\chi = -2$ and -4 (and hence genus 2 and 3) was given by H. S. M. Coxeter and W. O. J. Moser [CMo84] and F. A. Sherk [She59], respectively. None of the maps for the two genera are chiral. Also, regular maps of genus 2 turn out to have no antipodal reflections, implying that there are no regular maps on the nonorientable surface with $\chi = -1$. Using similar methods, A. S. Grek in a series of papers [Gre63, Gr66a, Gr66b] derived a classification of regular maps on non-orientable surfaces with $-2 \geq \chi \geq -4$.

A more powerful method based on permutation representations is due to D. Garbe [Gar69], introduced in the course of classification of regular maps on the orientable surface of genus 4. Suppose that one wants to classify regular maps of type $\{m, k\}$ with exactly d faces. This is equivalent to classifying all $(k, m, 2)$-groups G of order $2md$. Since G contains a dihedral subgroup H of order $2m$, we have a permutation representation of G of degree d given by the action of G on the cosets of H; the image of H in this representation is the stabiliser of an element. Now, let $T = \langle (x, y, z), (yz)^k = (zx)^m = 1 \rangle$ be the full $(k, m, 2)$-triangle group and let N be the torsion-free normal subgroup of T such that $T/N \cong G$. Because of absence of torsion, the image of the dihedral group $L = \langle z, x \rangle$ under the natural projection $\theta : T \to T/N$ is again a dihedral group of order $2m$; in fact, we may assume that the image is H. We do not know G and N yet, but observe that the transitive permutation representation of G of degree d lifts onto a transitive permutation representation of T of the same degree d but this time on the cosets of the subgroup $\theta^{-1}(H)$. This suggests the following algorithm to determine all such groups G.

(A) Construct all transitive permutation representations $\psi : T \to S_d$ where S_d is the symmetric group of degree d acting on the set $\{1, 2, \ldots, d\}$, such that $\psi(L) = \mathrm{Stab}(1)$, the stabiliser of the element 1 in the image $\psi(T)$.

(B) Construct all epimorphisms ϑ from the subgroup $K = \psi^{-1}(\mathrm{Stab}(1)) < T$ onto the dihedral group D_m of order $2m$, such that $\vartheta(L) = D_m$ and such that $N = \ker(\vartheta)$ is torsion-free.

Then, clearly, $G = T/N$ is a $(k,m,2)$-group and the corresponding regular map has exactly d faces, that is, $|G| = 2md$. Indeed, an easy calculation shows that $|G| = |T/N| = [T : N] = [T : K][K : N] = [\psi(T) : \psi(K)]|D_m| = [\psi(T) : \mathrm{Stab}(1)] \cdot 2m = 2md$, as claimed. The fact that this algorithm constructs *all* regular maps of type $\{m,k\}$ with exactly d faces follows from the above discussion. In addition, all such *orientable* maps are filtered out by the condition that N be a subgroup of the orientation preserving part of T, that is, $N < \langle r,s \rangle$ where $r = yz$ and $s = zx$. Running the procedure with T, L, and D_m replaced by the $(k,m,2)$-triangle group $\langle r,s |\ r^k = s^m = (rs)^2 = 1\rangle$, the cyclic group $\langle s \rangle$, and the cyclic group Z_m, respectively, and identifying the normal subgroups N of T that are *not* normal in the *full* $(k,m,2)$-triangle group, one obtains all the chiral maps with d faces.

In practice, for each permutation representation in (A) one finds with the help of the Reidemeister-Schreier method a presentation of K and then one searches over the epimorphisms in (B). As long as the number d of faces is relatively small, the calculations – although time consuming and far from trivial – can be done by hand. This was the main tool used in the classification of regular and chiral maps on orientable surfaces of genus 5 and 6 (P. Bergau and D. Garbe [BeG89]), and 7 (D. Garbe in [Gar78]). The corresponding nonorientable results were obtained for Euler characteristic -5 by J. Scherwa [Sch85] and for -6 by P. Bergau and D. Garbe [BeG89]. In this connection it is worth noting that S. E. Wilson [Wi78b] has outlined a similar algorithm using a geometric language.

Summing up, by the late 1980's, the collective effort of the researchers mentioned above resulted in classification of all regular and chiral maps on orientable surfaces of Euler characteristic $\chi \geq -12$ (that is, up to genus 7), and regular maps on nonorientable surfaces with $\chi \geq -6$ (up to genus 8). It is interesting to note that, in the orientable case, there are no chiral maps of orientable genus between 2 and 6 at all! Further progress came in about a decade, when M. Conder and P. Dobcsányi [CoD01] published a computer-assisted classification of all regular and chiral maps on orientable surfaces up to genus 15, and regular maps on nonorientable surfaces up to genus 30. The authors used their own adaptation of the low-index subgroup algorithm and applied it to finding 'small' index normal subgroups – but not subgroups of the full triangle groups. Instead, it turned out to be of advantage to consider normal subgroups of the group $\langle x,y,z |\ x^2 = y^2 = z^2 = (xy)^2 = 1\rangle$, with the relator $(zx)^3$ or $(zx)^4$ added in the case when $m = 3$ or $m = 4$, and extract the rest from there. Re-confirming all the earlier results, the list in [CoD01] documents the state-of -the-art of the regular maps classification problem at the end of the millenium. In particular, by then a complete classification was known only for a finite number of surfaces.

5 Regular Maps on Surfaces of Large Genus

A breakthrough in classification of regular maps was achieved when A. Breda d'Azevedo, R. Nedela and the author [BNS05] classified regular maps on *all* surfaces of negative prime Euler characteristic – that is, on an *infinite* number of surfaces. Let $n(p)$ denote the number of regular maps, up to isomorphism and duality, on a surface with Euler characteristic $-p$, where p is a prime. The numbers $n(p)$ for $p < 29$ together with the automorphism groups of the maps have been determined by the computer-aided classification of [CoD01] and we therefore state the result of [BNS05] for the larger primes only. Also, for $p \equiv -1 \pmod 4$ let $\nu(p)$ be the number of pairs (j, l) such that $j > l \geq 3$, both j and l are odd, coprime, and $(j-1)(l-1) = p+1$.

Theorem 5.1 ([BNS05]). *Let p be an odd prime, $p \geq 29$ Then, up to isomorphism and duality, for the number $n(p)$ of regular maps on a surface with Euler characteristic $-p$ and for the corresponding groups G we have:*

(A) $n(p) = 0$ *if* $p \equiv 1 \pmod{12}$;
(B) $n(p) = 1$ *if* $p \equiv 5 \pmod{12}$, *with* $G \cong G_p = \langle r, s | \ r^{p+4} = s^4 = (rs)^2 = sr^3s^{-1}r^3 = 1\rangle$;
(C) $n(p) = \nu(p)$ *if* $p \equiv 7 \pmod{12}$, *with* $G \cong G_{j,l} = \langle r, s | \ r^{2j} = s^{2l} = (rs)^2 = (rs^{-1})^2 = 1\rangle$;
(D) $n(p) = \nu(p) + 1$ *if* $p \equiv -1 \pmod{12}$; *the groups G are G_p and the $\nu(p)$ groups $G_{j,l}$.*

Presentations of the above groups are given in terms of $r = yz$ and $s = zx$ only; recall that for nonorientable maps the automorphism group os generated by r and s. This implicitly assumes that the involutions x, y, z can be recovered from r and s in a unique way. While this is not true in general, in our case it can be shown that the involutions are indeed unique. We note that the group G_p is an extension of $Z_{(p+4)/3}$ by S_4 while for the groups $G_{j,l}$ we have $G_{j,l} \simeq D_j \times D_l$.

An important ingredient of the proof of Theorem 5.1 is the following milestone result in the project of classification of finite simple groups, due to D. Gorenstein and J.H. Walter [GoW65]: *If H is a group with a dihedral Sylow 2-subgroup and if $O(H)$ is the (unique) maximal normal subgroup of G of odd order, then $H/O(H)$ is isomorphic to either a Sylow 2-subgroup of G, or to the alternating group A_7, or to a subgroup of* $\mathrm{Aut}(PSL(2,q))$ *containing $PSL(2,q)$, where q is and odd prime power.* The original proof has about 180 pages and depends on the celebrated Feit-Thompson odd-order groups theorem. Subsequently, a shorter (but by no means simpler) proof of the Gorenstein-Walter result was given in [Ben81, BGl81] but it still depends on the odd-order groups theorem. An interesting challenge would be to find a proof of Theorem 5.1 without invoking such high-calibre results.

Every orientable surface carries a regular map; an example for any $g \geq 1$ is a single-face regular embedding of a bouquet of $2g$ circles. While M. Conder and B. Everitt [CoE95] showed that more than three quarters of nonorientable surfaces support a regular map, there are “gaps”. Absence of regular maps on nonorientable surfaces with Euler characteristic $\chi = 0$ (the Klein bottle) and $\chi = -1$ has been known for a long time. The list of [CoD01] shows that there are no regular maps on nonorientable surfaces with $\chi = -16$, -22, and -25, either.

From a deep study by S. Wilson and A. Breda [WBr04] it follows that nonexistence of regular maps extends also to $\chi = -37$ and -46 and that all the above are the only gaps in the range $-50 \le \chi \le 0$. The amazing consequence of Theorem 5.1 is that *there are infinitely many gaps*! Specifically, part (A) of the theorem and the list of [CoD01] imply that there are no regular maps on surfaces of Euler characteristic $\chi = -p$ where p is a prime congruent to 1 (mod 12) and $p \neq 13$.

Although a number of steps in the proof of Theorem 5.1 substantially depended on the primality of $-\chi$, extensions to small odd multiples of primes are within reach. A very recent work by G. A. Jones, R. Nedela and the author [JNS05] has resulted in a classification of regular maps on (nonorientable) surfaces with $\chi = -3p$ for sufficiently large primes.

A complete classification of regular or chiral maps on an infinite family of *orientable* surfaces is still not available. Nevertheless, important contributions to the study of automorphism groups of Riemann surfaces by M. Belolipetsky and G. A. Jones [BeJ04] imply ingredients for a classification of the regular and chiral maps on orientable surfaces of genus $p+1$ with ‘large’ automrophism group G in the sense that $|G| > 12p$ for the regular case, and $|G| > 6p$ for the chiral case, where p is a prime. In order to see what this means for regular maps, observe that the Euler characteristic of such a surface is $\chi = -2p$. From (2) we then obtain $|G| = -2\chi/(1/2 - 1/k - 1/m) > -4\chi = 8p$. In general, however, one may have regular maps with groups of order between $8p$ and $12p$ that are not captured by the results of [BeJ04].

A fair amount of structural information about automorphism groups of regular maps on a given surface can be extracted from Euler's formula (2) by just arithmetic considerations. An example is the following lemma due to M. Conder, P. Potočnik and the author [CPS04], which overlaps with an observation made by G. A. Jones [Jon04].

Lemma 5.2. *Let G be the automorphism group of a regular map on a surface with Euler characteristic $\chi \neq 0$ and let p' be a prime divisor of $|G|$ coprime with χ. Then, the Sylow p'-subgroups of G are cyclic if p' is odd, or dihedral if $p' = 2$. In particular, the Sylow 2-subgroups of G are automatically dihedral if χ is odd.*

This opens up the possibility of application of the Gorenstein-Walter theorem in the study of regular maps on surfaces with any odd Euler characteristic (which are necessarily nonorientable). Unfortunately, one still needs more information to determine the exact structure of the groups. Inspired by the first part of Lemma 5.2 it makes sense to look for cyclicity of the odd-order Sylow subgroups. And indeed, we have a good example at hand: Revisiting the proof of Theorem 5.1, it is easy to see that the claim (i) immediately implies that *all* the odd-order Sylow subgroups of the groups of regular maps on surfaces of Euler characteristic $\chi = -p$, p an odd prime, are cyclic! The same happens to be true for the groups of regular maps on surfaces with $\chi = -3p$ except for the groups $D_j \times D_l$ when $(j, l) = 3$. Another substantial contribution here is a further result of [CPS04]:

Proposition 5.3. *Let G be the automorphism group of a regular map on a surface with Euler characteristic $-p^2$ where p is an odd prime and $p \neq 3$. Then, every odd-order Sylow subgroup of G is cyclic.*

We see that there is more than enough motivation to investigate $(k, m, 2)$-groups G such that the Sylow 2-subgroups of G are dihedral and *all* the odd-order Sylow subgroups of G are cyclic. Such a class of groups in general has appeared in the literature in connection with various questions in group theory. Determination of groups with the property that all its Sylow subgroups (including the Sylow 2-subgroups) are cyclic goes back to W. Burnside and even to G. Frobenius (see [Bur11]). Relaxations of the Sylow 2-subgroups condition, however, turns the problem to a challenge which has been resolved only in special cases. We will say that H is a group of *Zassenhaus type* if all the odd-order Sylow subgroups of H are cyclic and if every Sylow 2-subgroup of H contains a cyclic subgroup of index 2. All the solvable groups of Zassenhaus type were determined by H. Zassenhaus [Zas36] and the characterisation of all such non-solvable groups was completed by M. Suzuki [Suz55] and W. J. Wong [Won66].

If G is a $(k, m, 2)$-group with presentation (1), the regular map $M = M(G; x, y, z)$ will be called a map of *Zassenhaus type* if the group G is of Zassenhaus type. In a recent work of M. Conder, P. Potočnik and the author, the above results together with the Gorenstein-Walter theorem have been used to classify all regular maps of Zassenhaus type. Since details of the classification are a little too long to be reproduced here, we just give a brief summary about the groups. In the case when the group G is solvable, the maps fall into 8 classes and G is either a dihedral group, or a split extension of a cyclic group by the Klein four-group, or else a split extension of the Klein four-group by a dihedral group (with certain congruence restrictions on the orders). If M is a map of Zassenhaus type with a non-solvable group G, then either $G \cong PGL(2, q)$ for some prime q (not a non-trivial power of a prime), or G is an extension of such a $PGL(2, q)$ by a cyclic group of order coprime with $q(q^2 - 1)$. As an aside, regular maps with automorphism groups isomorphic to projective (special as well as general) linear two-dimensional groups have been classified in great detail by the same set of authors in [CPS05].

These results are solid tools for investigation of regular maps of a given type $\{m, k\}$ on a surface with a given Euler characteristic χ. A nice corollary is a classification of regular maps of type $\{m, k\}$ on surfaces with $\chi = -p^2$

where p is a prime not dividing both k and m, such that $p \geq 23$; we refer to [CPS04] for particulars.

6 Conclusion: Regular Hypermaps

A natural avenue of research is to extend the classification to hypermaps. At the group-theoretic level, a regular (k, m, l)-*hypermap* is simply a finite group G generated by three involutions x, y, z in which the products yz, zx, and xy have orders k, m and l, respectively. Regular maps are therefore a special case of regular (k, m, l)-hypermaps for $l = 2$. Orientably regular and chiral hypermaps are defined analogously as in the case of maps. The theory of regular and chiral hypermaps is similar to the theory of regular and chiral maps, including connections with hyperbolic geometry and complex functions. Pictorial representations, however, are a little different and not unique. A topological representation of a hypermap that gives equal status to hypervertices, hyperedges, and hyperfaces (reflecting the equal status of the products yz, zx, and xy in the corresponding triangle group) can be obtained by means of an embedding of a trivalent graphs with faces of lengths $2k$, $2m$ and $2l$, alternating at each vertex.

The classification of regular maps on surfaces of negative prime Euler characteristic [BNS05] was extended to regular hypermaps by G. A. Jones [Jon04]. It turns out that the classification is obtained from the list of [BNS05] by adding just two items, the regular $(4, 3, 3)$- and $(6, 5, 4)$-hypermaps with groups $PSL(2, 7)$ and $PGL(2, 5) \cong S_5$. It is also worth mentioning that the deep study of gaps in the nonorientable range of Euler characteristics χ for $-50 \leq \chi < 0$ by S. Wilson and A. Breda [WBr04], which we have mentioned in the previous section, was actually done for regular hypermaps in general.

A combination of the approaches outlined in this brief survey, perhaps blended with the theory of linear and permutation representations of groups and with the general knowledge on group actions on compact surfaces, is likely to yield substantial contributions to the theory regular maps on a fixed surface in the indicated directions.

References

[BeJ04] M. Belolipetsky, G. A. Jones, Automorphism groups of Riemann surfaces of genus $p + 1$, where p is a prime, Submitted (2004).

[Ben81] H. Bender, Finite groups with dihedral Sylow 2-subgroups, *J. of Algebra* **70** (1981), 216–228.

[BGl81] H. Bender, G. Glauberman, Characters of finite groups with dihedral Sylow 2-subgroups, *J. Algebra* **70** (1981), 200–215.

[BeG89] P. Bergau, D. Garbe, Non-orientable and orientable regular maps, in *Proceedings of "Groups-Korea 1988"*, *Lect. Notes Math.* **1398**, Springer (1989), 29–42.

[Big71] N. L. Biggs, Automorphisms of imbedded graphs, *J. Combinat. Theory Ser. B* **11** (1971), 132–138.
[Bra26] H. R. Brahana, Regular maps on an anchor ring, *Amer. J. Math.* **48** (1926), 225–240.
[Bra27] H. R. Brahana, Regular maps and their groups, *Amer. J. Math.* **49** (1927), 268–284.
[BJN05] A. Breda d'Azevedo, G. A. Jones, R. Nedela, M. Škoviera, Chirality groups of maps and hypermaps, submitted (2005).
[BNS05] A. Breda d'Azevedo, R. Nedela, J. Širáň, Classification of regular maps of negative prime Euler characteristic, *Trans. Amer. Math. Soc.* **357** No. 10 (2005), 4175–4190.
[BrS85] R. P. Bryant, D. Singerman, Foundations of the theory of maps on surfaces with boundary, *Quart. J. Math. Oxford* Ser. (2) **36** No. 141 (1985), 17–41.
[Bur11] W. Burnside, *Theory of Groups of Finite Order*, Cambridge Univ. Press, 1911.
[CoD01] M. D. E. Conder, P. Dobcsányi, Determination of all regular maps of small genus, *J. Combinat. Theory Ser. B* **81** (2001), 224–242.
[CoE95] M. D. E. Conder, B. Everitt, Regular maps on non-orientable surfaces, *Geom. Dedicata* **56** (1995), 209–219.
[CPS04] M. D. E. Conder, P. Potočnik, J. Širáň, Regular maps on groups with cyclic odd-Sylow subgroups, preprint (2004).
[CPS05] M. D. E. Conder, P. Potočnik, J. Širáň, Regular hypermaps over projective linear groups, preprint (2005).
[Cox48] H. S. M. Coxeter, Configurations and maps, *Rep. Math. Colloq (2)* **8** (1948), 18–38.
[CMo84] H. S. M. Coxeter, W. O. J. Moser, *Generators and Relations for Discrete Groups*, 4th Ed., Springer-Verlag, Berlin 1984.
[DKN04] S. F. Du, J. H. Kwak, R. Nedela, Regular embeddings of complete multipartite graphs, submitted (2004).
[DKN05] S. F. Du, J. H. Kwak, R. Nedela, Classification of regular embeddings of hypercubes of odd dimension, submitted (2005).
[Dyc80] W. Dyck, Über Aufstellung und Untersuchung von Gruppe und Irrationalität regularer Riemannscher Flächen, *Math. Ann.* **17** (1880), 473–508.
[Gar69] D. Garbe, Über die regulären Zerlegungen geschlossener orientierbarer Flächen, *J. Reine Angew. Math.* **237** (1969), 39–55.
[Gar78] D. Garbe, A remark on non-symmetric Riemann surfaces, *Arch. Math.* **30** (1978), 435–437.
[GNS99] A. Gardiner, R. Nedela, J. Širáň, M. Škoviera, Characterization of graphs which underlie regular maps on closed surfaces, *J. London Math. Soc. (2)* **59** No. 1 (1999), 100–108.
[GoW65] D. Gorenstein, J. H. Walter, The characterization of finite groups with dihedral Sylow 2-subgroups, I, II, III, *J. of Algebra* **2** (1965), 85–151, 218–270, 334–393.
[Gre63] A. S. Grek, Reqular polyhedra of simplest hyperbolic types, (Russian) *Ivanov. Gos. Ped. Inst. Učen. Zap.* **34** (1963), 27–30.
[Gr66a] A. S. Grek, Regular polyhedrons on surfaces with Euler characteristic $\chi = -4$, (Russian) *Soobšč. Akad. Nauk Gruzin. SSR* **42** (1966), 11–15.
[Gr66b] A. S. Grek, Regular polyhedra on a closed surface with the Euler characteristic $\chi = -3$, (Russian) *Izv. Vysš. Učebn. Zaved. Matematika* **55** No. 6 (1966), 50–53.

[Gro84] A. Grothendieck, Esquisse d'un programme, preprint, Montpellier (1984).
[GrT87] J.L. Gross, T.W. Tucker, *Topological Graph Theory*, Wiley (1987) and Dover (2001).
[Hea90] P.J. Heawood, Map-colour theorem. Quart, *J. Math.* **24** (1890), 332–338.
[Hef98] L. Heffter, Über metazyklische Gruppen und Nachbarconfigurationen, *Math. Ann.* **50** (1898), 261–268.
[JaJ85] L.D. James, G.A. Jones, Regular orientable imbeddings of complete graphs, *J. Combinat. Theory Ser. B* **39** (1985), 353–367.
[Jon97] G.A. Jones, Maps on surfaces and Galois groups, *Math. Slovaca* **47** (1997), 1–33.
[Jon04] G.A. Jones, Large groups of automorphisms of compact surfaces with prime negative Euler characteristic, preprint (2004).
[JNS05] G.A. Jones, R. Nedela, J. Širáň, Classification of regular maps of Euler characteristic $-3p$, where p is a prime, preprint (2005).
[JNS04] G.A. Jones, R. Nedela, M. Škoviera, Regular embeddings of $K_{n,n}$ where n is odd prime power, preprint (2004).
[JoS78] G.A. Jones, D. Singerman, Theory of maps on orientable surfaces, *Proc. London Math. Soc.* (3) **37** (1978), 273–307.
[JoS96] G.A. Jones, D. Singerman, Belyĭ functions, hypermaps, and Galois groups, *Bull. London Math. Soc.* **28** (1996), 561–590.
[Kep19] J. Kepler, *The harmony of the world*, (Translation from the Latin "Harmonice Mundi", 1619) *Memoirs Amer. Philos. Soc.* **209**, American Philosophical Society, Philadelphia, PA (1997).
[Kle79] F. Klein, Über die Transformation siebenter Ordnung der elliptischen Functionen, *Math. Ann.* **14** (1879), 428–471.
[KwK05] J.H. Kwak, Y.S. Kwon, Regular orientable embeddings of complete bipartite graphs, preprint (2005).
[Sah69] C.H. Sah, Groups related to compact Riemann surfaces, *Acta Math.* **123** (1969), 13–42.
[Sch85] J. Scherwa, *Regulaere Karten geschlossener nichtorientierbarer Flaechen*, Diploma Thesis, Bielefeld (1985).
[She59] F.A. Sherk, The regular maps on a surface of genus three, *Canad. J. Math.* **11** (1959), 452–480.
[Suz55] M. Suzuki, On finite groups with cyclic Sylow subgroups for all odd primes, *Amer. J. Math.* **77** (1955), 657–691.
[Tuc83] T.W. Tucker, Finite groups acting on surfaces and the genus of a group, *J. Combinat. Theory Ser. B* **34** No. 1 (1983), 82–98.
[Tut73] W.T. Tutte, What is a map? In *New Directions in Graph Theory* (F. Harary, Ed.), Acad. Press (1973), 309–325.
[Wi78a] S.E. Wilson, Non-orientable regular maps, *Ars Combin.* **5** (1978), 213–218.
[Wi78b] S.E. Wilson, Riemann surfaces over regular maps, *Canad. J. Math.* **30** (1978), 763–782.
[WBr04] S.E. Wilson, A. Breda D'Azevedo, Surfaces having no regular hypermaps, *Discrete Math.* **277** (2004), 241–274.
[Won66] W.J. Wong, On finite groups with semi-dihedral Sylow 2-subgroups, *J. Algebra* **4** (1966), 52–63.
[Zas36] H. Zassenhaus, Über endliche Fastkörper, *Abh. Math. Sem. Univ. Hamburg* **11** (1936), 187–220.