Regular Maps and Their Groups.

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The geometrical representation of a finite group by means of fundamental regions, due to Dyck,* leads to a map that is transformed into itself by every operation of the group. The group may be represented transitively on symbols for regions, but in general it may not be represented transitively on symbols for edges or vertices. On the other hand it is well known † that the polyhedral groups may be represented transitively on symbols for regions, for edges, or for vertices of the regular polyhedra or the corresponding maps on a sphere. In the one case the number of regions of the map is equal to twice the order of the group and in the other the number of regions is equal to the order of the group divided by the number of sides of a region. Maps of the latter type we have called regular maps.

The polyhedral groups are the only ones that may be represented on regular maps on a sphere. There has been no attempt to determine the kinds of groups that may be represented on regular maps on surfaces of higher genus. Heffter ‡ showed that the metacyclic groups may be so represented. In a recent paper § all the maps of twelve five-sided regions with a group of order 120 containing an icosahedral subgroup were exhibited. In this last cited investigation the surfaces were allowed to be one-sided or two-sided and when the surface was two-sided a transformation of the map into itself in such a way as to reverse the sense of the boundary of each region was permitted. Lastly, the maps on a surface of genus one and the groups connected with them were considered.* It was shown that any such regular map must be made up of triangles, of quadrangles, or of hexagons and their groups must be generated by two operators of orders two and three with product of order

* "Gruppentheoretischen Studien," *Mathematische Annalen*, Vol. 20 (1882), pp. 1-44.

† Klein, *Vorlesungen ueber das Ikosaeder*, I, § 13.

‡ "Ueber metacyklische Gruppen und Nachbarconfigurationen," *Mathematische Annalen*, Vol. 50 (1897), pp. 261-268.

§ Brahana and Coble, "Maps of twelve countries, etc.", *American Journal of Mathematics*, Vol. 48 (1926), pp. 1-20.

* "Regular Maps on an Anchor Ring," *American Journal of Mathematics*, Vol. 48 (1926), pp. 225-240.

six, or by two operators of order two and four with product of order four, and conversely that to every such group there corresponds a regular map on an anchor ring.

In § 1 we recall a condition that has been stated previously as necessary that a given group be the group of a regular map and prove that it is sufficient by giving a method of constructing the map corresponding to the given group. In § 2 we note four classes of groups that satisfy the given condition and determine the genera of the surfaces on which the corresponding maps lie. In § 3 we apply the earlier considerations to determine the regular maps on a surface of genus two.

1. *The Group of a Regular Map.* A *map* is a finite set of distinct 0-cells, 1-cells, and 2-cells which constitutes a closed, connected, two-sided, two-dimensional manifold. We shall hereafter use the terms *vertices, edges* or *lines,* and *regions* for 0-cells, 1-cells, and 2-cells.

If the interior of a region of the map is imaged continuously on the interior of a circle there will be a finite * set of points on the boundary of the circle which correspond to the vertices of the map that lie on the boundary of the given region. The number of points in this set is the number of *vertices of the region.* The number of vertices of a region is the same as the number of *sides or edges of the region.*

A map of k n-sided regions will be said to be *regular* if the number of ways in which the surface can be put into $(1-1)$ correspondence with itself so that regions correspond to regions, edges to edges, and vertices to vertices, without reversal † of the sense of the boundaries of regions is kn.

The operations of transforming a map into itself in such a way as to preserve the sense of a region obviously constitute a group; this group will be called *the group of the map.* A map of k n-sided regions will be regular if its group is of order kn.

It is immediately evident and has been noted otherwhere * that the group of a regular map may be generated by two operators, viz.: S which leaves a given region fixed permuting its edges cyclically, and T which leaves an edge of this region fixed interchanging the two regions which have the edge in

* We exclude the case of a map defined by designating a single point on a sphere as the boundary of the remainder of the surface.

† It is well-known that the regions of a map on a two-sided surface may be sensed alike in two distinct ways, cf. Veblen and Young, *Projective Geometry*, Vol. II, p. 495. We require that there be kn ways of transforming k positively sensed regions into the same k positively sensed regions.

* Brahana and Coble, *loc. cit.*, p. 5.

common. T is of order two. By means of these two operations and combinations of them any region of the map may be transformed into any other region with a particular vertex of the first going into any vertex of the second.

In demonstrating the sufficiency of the above condition, we shall distinguish two cases: (1) T is not permutable with S or any group generated by a power of S; and (2) there exists a group generated by a power of S which is permutable with T. The tetrahedral, octahedral, and icosahedrol groups come within the first category and the dihedral groups are in the second.

We consider a group G generated by two operators S and T, the latter of order two and not permutable with any group generated by a power of the former. We distribute the operators of G in k right co-sets with respect to the subgroup H consisting of S and its powers and denote each co-set by a letter. Multiplication of all the operators of G on the right by any operator of G interchanges the co-sets and so determines a substitution on the k letters. The resulting substitution group will be transitive and will be simply isomorphic with G. We note for future use that the only substitutions which omit the letter corresponding to H are those which correspond to operators of H.

Let S_a, T_{ab}, and S_b be the substitutions corresponding to S, T, and TST respectively, b being the letter into which a is transformed by T_{ab}. The letter b appears in a cycle of S_a, otherwise S_b would omit a contrary to the hypothesis that T is not permutable with any subgroup generated by a power of S. The cycle C of S_a which contains b is of degree n, otherwise some power of S_a lower than the nth would leave b fixed; this would imply that $ST \cdot S^m = S^l T$ or $S^m = TS^{l-1}T$ which is impossible for the same reason.

We may now construct a regular map corresponding to G. A polygon of n sides may be denoted by a and bounded cyclically by n n-sided polygons named from C. The transform of a and C by T_{ab} gives b and its boundary. Continuing this process we get k bounded n-sided polygons. Since there are n substitutions leaving a fixed there will be n substitutions leaving b fixed and b will appear n times in the conjugates of C. Therefore, each region appears on the boundaries of n other regions. We may join the k polygons into a simply connected polygon in a plane with its edges paired in the ordinary manner. The two dimensional manifold so defined is two-sided, for any operation leaving a fixed is a power of S_a and so leaves C unchanged making it impossible to transform a into itself with its boundary reversed. Hence, *to every group in the first class there corresponds a regular map.*

The groups of the second category will be examined in two distinct classes according as (a) T is permutable with S, or (b) T is not permutable with S but is permutable with a group generated by a power of S.

The groups of class (a) are cyclic and of even order if T is a power of S. A map corresponding to such a group is obtained by taking a polygon of n sides and joining opposite sides so as to give a two-sided surface.* An example of such a map is a four-sided region on an anchor ring; it contains one region, one vertex, and two edges.

If T is not a power of S the group is Abelian and of order $2n$. If there exists a regular map corresponding to it and containing an n-sided region it must contain two such regions. We may take two n-sided polygons and letter their sides $(abc \cdots f)$ and $(\alpha\beta\gamma \cdots \zeta)$ respectively. We join them toether along a and α, making b correspond to β, c to γ, and so on, the members of a pair being oppositely sensed with respect to the double polygon. Such a map admits the operations $(abc \cdots f)(\alpha\beta\gamma \cdots \zeta)$ and $(a\alpha)(b\beta)(c\gamma) \cdots (f\zeta)$. It may be readily verified that the number of vertices is 1 or 2 according as n is odd or even and that the genus of the resulting surface is $(n-1)/2$ or $(n-2)/2$.

In the groups of class (b) the subgroup $1, T$ is not invariant and we may represent the group as a substitution group on symbols for the co-sets with respect to this subgroup by the method used for groups of the first category. If a is the letter corresponding to the set $1, T$ the substitution corresponding to S contains a in a cycle C of n letters, for the n co-sets $\begin{matrix} 1 & S & S^2 & \cdots S^{n-1} \\ T & TS & TS^2 & \cdots TS^{n-1} \end{matrix}$ are all distinct. If we denote the order of G by kn as before, we see that the substitution group is of degree $kn/2$. The number of letters in C and its conjugates is kn so that each of the letters appears twice. We may construct a map corresponding to G by taking an n-sided polygon for each of the conjugates of C and bounding it according to the letters of the conjugate. This time we name edges on the boundary instead of regions across the boundary. The polygons may be joined into a single two-sided surface by coalescing like-named edges with the usual precautions as to the senses of corresponding edges.

We note that the dihedral groups are contained in this class. If T transforms S into its inverse the group is dihedral. The resulting map contains two n-sided regions and lies on a sphere. It may be obtained by drawing an

* The only requirement is that the pair of corresponding sides a and a' be oppositely sensed on the boundary of the polygon and that they be then joined so that the two senses coincide. See Brahana, "Systems of Circuits on Two-Dimensional Manifolds," *Annals of Math.*, Vol. 23 (1922), p. 146.

n-sided polygon on a sphere. T may transform S into some power of itself other than its inverse in which case the group is again of order $2n$ and the map consists of two n-sided polygons not on a sphere. If neither of the above conditions obtain the map still resembles the dihedral maps in that each region touches some other region along more than one edge. To see this let S_α and S_β be generators of the groups leaving the neighboring regions α and β fixed, and let $T_{\alpha\beta}\, S_\alpha\, T_{\alpha\beta} = S_\beta$, where S_α and $T_{\alpha\beta}$ are the substitutions corresponding to S and T respectively. Since $T\, S^m\, T = (S^m)^r$ for some m less than n, then $T_{\alpha\beta}\, S_\alpha{}^m\, T_{\alpha\beta} = (S_\alpha{}^m)^r = (S_\beta)^m$ leaves both α and β fixed and since $mr \neq n$ S_α must transform an edge common to α and β into another edge common to α and β.

We return from the digression of the last paragraph to state the principal result so far obtained in the following theorem:

A necessary and sufficient condition that G be the group of a regular map is that G be generated by two operators of which one is of order two.

2. *Some Types of Group that Give Regular Maps. The Symmetric and Alternating Groups.* It is well known that the tetrahedral, octahedral, and icosahedral groups are simply isomorphic with the alternating and symmetric groups of degree four and the alternating group of degree five. We extend the above result by means of two theorems, of which the first is:

To the symmetric group of degree n there corresponds a regular map of $(n-1)!$ *n-sided regions on a surface of genus*

$$p = 1 + (n-2)!\ (n^2 - 5n + 2)/4.$$

The existence of the map follows from the theorem of §1 and a theorem due to Moore * that the symmetric group of degree n is generated by two operators of orders n and two whose product is of order $(n-1)$. In order that we may determine the genus of the surface on which the map lies we shall recall a theorem that was used in *Regular Maps on an Anchor Ring.* In that paper it was proved (p. 227), though not explicitly stated, that

If S generates the group leaving a region fixed and T is the operator leaving an edge of the same region fixed, then ST generates the group leaving a vertex of the region fixed.

* *Proceedings of the London Mathematical Soc.,* Vol. 28 (1896), pp. 357-366. See also Carmichael, *Quarterly Journal,* Vol. 49 (1922), p. 235.

From this theorem is follows that the number of regions at a vertex in a regular map is equal to the order of ST. In the maps under consideration there are thus $(n-1)$ regions at a vertex. The number of vertices is $n(n-2)\,!$, the number of edges is $n\,!/2$, and the number of regions is $(n-1)\,!$. From the Euler formula we obtain the genus $p=1+(n-2)\,!\,(n^2-5n+2)/4$. We give in the following table a list of the maps described by the theorem for $n=4$, 5, 6, and 7.

n	k	p	n	k	p
4	6	0	6	120	49
5	24	4	7	720	481

The first of these maps is the cube; the second is one of the maps described in *Maps of Twelve Five-Sided Regions*, etc. (*l. c.*, p. 19) and is the doubly covered figure II of that paper.

The alternating groups of degree greater than three are among the groups that give regular maps, for each group is generated by two operators one of which is of order two.* The generators may be chosen to be of orders 2 and $(n-1)$ with product of order $(n-1)$ if n is even, and of orders 2 and $(n-2)$ with product of order n if n is odd. When n is even we have $n(n-2)\,!/2$ vertices, $n\,!/4$ edges, and $n(n-2)\,!/2$ regions. The Euler formula takes the form

$$n(n-2)\,!/2-n\,!/4+n(n-2)\,!/2=2(1-p),$$

whence

$$p=1+n(n-2)\,!\,(n-5)/8.$$

When n is odd we have $(n-1)\,!/2$ vertices, $n\,!/4$ edges, and

$$n(n-1)(n-3)\,!/2$$

regions. Hence,

$$p=1+(n-1)(n-3)\,!\,(n^2-6n+4)/8.$$

We have the following theorem:

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* It may be shown readily that $S=(a_1, a_2, \cdots, a_{n-1})$ and $T=(a_1a_2)(a_3a_n)$ for n even, and $S=(a_1, a_2, \cdots, a_{n-2})$ and $T=(a_1a_{n-1})(a_2a_n)$ for n odd, generates the alternating group of degree n.

To the alternating group of degree $n(>3)$ there corresponds a regular map of $n(n-2)!/2$ $(n-1)$-sided regions or a map of $n(n-1)(n-3)!/2$ $(n-2)$-sided regions on a surface of genus $1+n(n-2)!(n-5)/8$ or $1+(n-1)(n-3!)(n^2-6n+4)/8$ according as n is even or odd.

It is of some interest to note the genera of the maps for small values of n. In the following list n' is the number of sides of each region of the map given by the theorem above.

n	n'	k	p	n	n'	k	p
4	3	4	0	6	5	72	19
5	3	20	0	7	5	504	199.

The first two are the tetrahedron and the icosahedron respectively.

Every regular map determines a second regular map which we shall call its *dual.* The dual of a map is obtained by taking a point within each region and joining the points of every pair of neighboring regions by an arc across their common edge, or by an arc across each common edge if more than one exists, the arcs being chosen so that no two intersect. The resulting map has a region for each vertex and a vertex for each region of the original map; the number of edges is the same in both. The cube and the octahedron are dual to each other, as are also the dodecahedron and the icosahedron. The dual of the tetrahedron is a tetrahedron; such a map will be called *self-dual.*

If S and T are the generators of the group from which a given map is obtained by the methods of § 1, the generators of the same group which would give the dual map are (ST) and T. The number of sides of a region of a map is equal to the number of regions at a vertex of its dual. Hence, *a necessary and sufficient condition that the map corresponding to the generators S and T be self-dual is that the orders of S and ST be the same.*

We have immediately the following theorem:

The maps given by the theorem concerning alternating groups are self-dual whenever n is even.

Subgroups of the Metacyclic Groups. The metacyclic group $G_{p(p-1)}$ of degree p (p must be prime) is generated by an element Σ of order p and a cyclic element S of order $(p-1)$.* The element Σ generates an invariant

* Netto, *Substitutiontheorie,* § 125.

subgroup and the remaining operators are the transforms of powers of S by Σ and its powers. S and Σ satisfy a relation $S^{-1}\Sigma S = \Sigma^m$, where $m^{p-1} \equiv 1$, mod p, and m satisfies no relation of the form $m^x \equiv 1$, mod p, where $x < p - 1$. Since p is a prime (> 2) S is of even order and $S^{(p-1)/2}$ is of order 2.

If $p - 1$ contains any even factor ρ and $p - 1 = \rho\lambda$ then S^λ generates a cyclic group of order ρ which contains an element of order two. S^λ and Σ generate a group of order ρp. We shall show that this group $G_{\rho p}$ may be generated by $S' = S^\lambda$ and any of its elements of order two except $(S')^{\rho/2}$, e. g. by $T' = \Sigma^{-k} S'^{\rho/2} \Sigma^k$. From the fact that $S^{-1}\Sigma S = \Sigma^m$, we have $\Sigma S = S\Sigma^m$, and $\Sigma S' = S' \Sigma^{m^\lambda}$. Hence

$$T' = \Sigma^{-k}(S')^{\rho/2}\Sigma^k = (S')^{\rho/2}\Sigma^{-km^{(p-1)/2}}\Sigma^k = (S')^{\rho/2}\Sigma^{2k}.$$

This last relation is due to the fact that

$$m^{p-1} - 1 = (m^{(p-1)/2} - 1)(m^{(p-1)/2} + 1) \equiv 0, \text{ mod } p.$$

The invariant subgroup is generated by any power of Σ except identity and hence $[S', T']$ contains Σ. Therefore to every group $G_{\rho p}$ there corresponds a regular map.

In order to determine the genus of the surface on which the map corresponding to $G_{\rho p}$ lies we must find the order of $S'T'$. From considerations similar to those used above we see that

$$S'T' = (S')^{[(\rho/2)+1]} \quad \Sigma^{2k}$$

$$(S'T')^2 = (S')^{2[(\rho/2)+1]} \quad \Sigma^{2k(m^{[(\rho/2)+1]\lambda}+1)}$$

$$= (S')^{2[(\rho/2)+1]} \quad \Sigma^{2k(-m^\lambda+1)}$$

$$(S'T')^3 = (S')^{3[(\rho/2)+1]} \quad \Sigma^{2k(m^{2\lambda}-m^\lambda+1)}$$

and in general,

$$(S'T')^n = (S')^{n[(\rho/2)+1]} \quad \Sigma^{2k(1+m^\lambda-m^{2\lambda}+\dots+(-1)^{n-1}m^{(n-1)\lambda})}$$

If we set $(S'T')^n = 1$ and seek the smallest value of n that will satisfy the relation, we shall have to find the smallest value of n that will satisfy the two congruences

(a) $\lambda n[(\rho/2) + 1] \equiv 0$, mod $p - 1$, and

(b) $2k(1 - m^\lambda + m^{2\lambda} - m^{3\lambda} + \cdots + (-1)^{n-1} m^{(n-1)\lambda}) \equiv 0$, mod p.

If n is odd (b) takes the form $\frac{1+m^{n\lambda}}{1+m^{\lambda}} \equiv 0$, mod p, the $2k$ being dropped because p is prime. This implies $m^{n\lambda} = -1$, mod p, hence $n = \rho/2$ satisfies the congruence provided $\rho/2$ is odd. This value also satisfies (a) since $(\rho/2)+1$ is even and $n[(\rho/2)+1]$ is a multiple of ρ. If n is even (b) takes the form $\frac{1-m^{n\lambda}}{1+m^{\lambda}} \equiv 0$, mod p, in which case $m^{n\lambda} \equiv 1$ mod p, and $n = \rho$. This value obviously satisfies (a). Therefore the order of $S'T'$ is ρ or $\rho/2$ according as $\rho/2$ is even or odd.

The corresponding map will have p regions, $p\rho/2$ edges, and p or $2p$ vertices. From the Euler formula we find the genus of the surface. We state the result in the following theorem:

To every group of even order $p\rho$ which is contained in the metacyclic group of degree p there corresponds a regular map of p ρ-sided regions on a surface of genus $1+(p/4)(\rho-4)$ or $1+(p/4)(\rho-6)$ according as $\rho/2$ is or is not a multiple of 4.

Since when ρ is a multiple of 4 S' and $S'T'$ are of the same order, we have

The maps corresponding to $G_{\rho p}$ are self-dual whenever ρ is a multiple of 4.

When $\rho = 2$ the groups are dihedral and the maps lie on a sphere. When $\rho = p-1$ the groups are the metacyclic groups themselves and the maps are those given by Heffter (cf. the reference above). When $\rho = 4$ or 6 the maps lie on an anchor ring (cf. above).

It is worthy of note that of all the maps whose existence we have proved very few lie on surfaces of low genus. The Euler formula $V - E + F = 2(1-p)$ shows that if p is to be small V and F must be as large as possible, which for a group of given order requires that n and v, the orders of S and ST, be small. If we seek a map on a surface of low genus whose group is a subgroup of a metacyclic group the degree of the group or ρ or both must be small. If the genus is to be greater than 1 p must be at least 11 and ρ must be at least 8. The metacyclic group of degree 11 gives a map on a surface of genus 12; the map on the surface of lowest genus corresponding to a group G_{8p} is made up of 17 octagons on a surface of genus 18. The smallest map of decagons corresponds to a G_{31p} and lies on a surface of genus 32. We note in passing that whenever p is of the form $40h+1$ there exists a map of p octagons and a map of p decagons each on a surface of genus $p+1$. The more interesting maps, i. e. those on surfaces of low genus, are all missing except some of the maps on an anchor ring.

Subgroups of the Modular Group. The two-rowed unit matrices with integer elements constitute a group simply isomorphic with the modular group.* If the elements of each matrix are reduced modulo n there is obtained a finite set of matrices of determinant 1, mod n, which constitute a group isomorphic $(1 - \infty)$ with the modular group. This group, $G_{2\mu(n)}$, contains a single element of order two, viz.: $\begin{pmatrix} n-1 & 0 \\ 0 & n-1 \end{pmatrix}$, and so is not available for the group of a regular map. If, however, we make the further reduction of considering $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ to be equivalent to $\begin{pmatrix} -a & -b \\ -c & -d \end{pmatrix}$ we obtain $G_{\mu(n)}$ which is in $(1-2)$ isomorphism with $G_{2\mu(n)}$. The group $G_{\mu(n)}$ is generated by the two operators $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 \\ n-1 & 0 \end{pmatrix}$ which we denote by S and T. The order of S is n, the order of T is 2, and the order of ST is 3. The order of the group is given by the equation

$$\mu(n) = (n^3/2)(1 - 1/q_1^2)(1 - 1/q_2^2) \cdots \quad \text{where } n = q_1^{\gamma_1} q_2^{\gamma_2} \cdots$$

and the q's are distinct primes.

The group $G_{\mu(n)}$ determines a map of $\mu(n)/n$ n-sided regions, $\mu(n)/2$ edges, and $\mu(n)/3$ vertices. Using these values in the Euler formula we determine the genus of the surface. The result is the following theorem:

To every group $G_{\mu(n)}$ there corresponds a regular map of $\mu(n)/n$ n-sided regions on a surface of genus $1 + (1/6 - 1/n)\mu(n)$.

It is evident that the maps associated by Klein-Fricke with the groups $G_{\mu(n)}$ have a close resemblance to the regular maps we have obtained. Their maps consist of $2\mu(n)$ triangles in which the subgroups of order n are represented by the $2n$ triangles that come together at a vertex of one type, the subgroups of order three by the six triangles at a vertex of another type, and the elements of order two by the four triangles at a vertex of a third type. We represent a subgroup of order n by a single n-sided region, which amounts to combining the $2n$ triangles at a vertex of the first type into a single region. If this combination is made at each of the vertices of the first type their

* The facts of this paragraph are to be found in Klein-Fricke, *Theorie der Elliptischen Modulfunktionen*, Leipzig (1890), Chapter 7.

maps will obviously become regular maps in the sense in which we are using the term.

For certain composite values of n $G_{\mu(n)}$ will contain distinct elements of the form $\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$; for example, in addition to $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ we have, when $n = 8$, $\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$ and, when $n = 15$, $\begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}$. These elements will constitute an invariant subgroup of $G_{\mu(n)}$ and if this subgroup is of order m we obtain a group $G_{\mu(n)}/m$ in $(1 - m)$ isomorphism with $G_{\mu(n)}$ by considering the operators of this invariant subgroup as equivalent. The groups $G_{\mu(n)}/m$ satisfy the condition of § 1 and so give regular maps.

The following is the list of maps corresponding to groups $G_{\mu(n)}$ for small values of n; it is essentially the combination of two lists given by Klein-Fricke.

k	n	p	k	n	p
4	3	0	24	7	3
6	4	0	24	8	5
12	5	0	36	9	10
12	6	1	36	10	13

The first four are regular polyhedra and a map on an anchor ring.

The subgroups of $G_{\mu(n)}$ for n prime are described by Klein-Fricke. Every such subgroup, and the metacyclic groups are among them, which is generated by two operators one of which is of order two gives a regular map. We shall not pursue this question further but shall note two groups of low order that give maps on surfaces of low genus. The group already mentioned obtained by taking $G_{\mu(8)}$ and considering $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$ to be equivalent gives a G_{96} which corresponds to a map of 12 octagons on a surface of genus 3. If we take $G_{\mu(8)}$ and consider $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 1 & 4 \\ 4 & 1 \end{pmatrix}$. $\begin{pmatrix} 3 & 4 \\ 0 & 3 \end{pmatrix}$ and $\begin{pmatrix} 3 & 0 \\ 4 & 3 \end{pmatrix}$ equivalent we obtain a G_{48} which corresponds to a map of 6 octagons on a surface of genus 2. (Klein-Fricke, p. 652.) To see that the map is made up of octagons we note that $S = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ is still of order 8, since $\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ implies $a \equiv 0$, mod 8. We note for a later reference that $S^4T = TS^4$, where $T = \begin{pmatrix} 0 & 1 \\ 7 & 0 \end{pmatrix}$, for $TS^4T = \begin{pmatrix} 0 & 1 \\ 7 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ 7 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 4 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 4 \\ 4 & 1 \end{pmatrix}$.

3. *The Regular Maps on a Surface of Genus Two.* Our method of investigating this question is first to determine from the Euler formula the possible maps and then to examine the possibility of the existence of groups having the characteristics required by the maps. From the maps we are able to determine the order of the group, the orders of two generating operators, and the order of their product. Since the existence of a map implies the existence of a dual we may choose the generating operators of the group in two distinct ways whenever the proposed map is not self-dual. This is of appreciable advantage whenever the map is such that the generating operators of the group may be chosen so that S is of prime order, for it assures us that the order of the group must be twice the order of S if T is permutable with a group generated by a power of S, or else, in the opposite case, the group may be represented on symbols for regions of the proper one of the two dual maps. Having the order and degree of the group we are able to make use of the work that has been done in listing groups of low degree.

We shall consider first the cases of maps of one and two regions respectively. For a map of one region on a surface of genus two the Euler formula takes the following form $(n/v) - (n/2) + 1 = -2$ where v is the number of regions at a vertex and n is the number of sides of a region. This equation may be written in the form $(n-6)(v-2) = 12$. $(v-2)$ is a positive integer, $(n-6)$ is therefore positive and is an integer because n is an integer. The possible solutions are

$$\begin{array}{llllll} n = 7, & 8, & 9, & 10, & 12, & 18 \\ v = 14, & 8, & 6, & 5. & 4, & 3. \end{array}$$

The sides of the region are to be joined in pairs and so n cannot be odd. If $n = 8$ S is of order 8 and $T = S^4$. Then $ST = S^5$ and is of order 8. Since this is the value of v it follows that *there exists a map of a single octagon on a surface of genus two.*

If $n = 10$ we have $S^{10} = 1$, $T = S^5$, and $ST = S^6$ is of order 5 which is the value of the corresponding v. Hence, *there exists a map of a single decagon on a surface of genus two.*

If $n = 12$ we have $S^{12} = 1$, $T = S^6$, and $ST = S^7$ is of order 12. This is not the proper value for v, and hence there is no map. If $n = 18$ we have $S^{18} = 1$, $T = S^9$, and $ST = S^{10}$ is of order 9. There is no map in this case.*

If the map contains two regions we get in the same manner as above $(n-4)(v-2) = 8$. The possible solutions are

* The last two groups give maps on surfaces of genera 3 and 4 respectively.

$$n = 5, \quad 6, \quad 8, \quad 12$$
$$v = 10, \quad 6, \quad 4, \quad 3.$$

The dual of the map of a single decagon above corresponds to the case $n = 5$. Thus, *there exists a map of two pentagons on a surface of genus two.* We note that the second map given above is self-dual and therefore we do not expect to find its dual again. Since the group of order 10 is either cyclic or dihedral and since the map corresponding to the latter group is on a sphere there is no other map of two pentagons on a surface of genus two.

If $n = 6$ we note that the group must be Abelian and S and T must be independent generators, for $TST = S^m$ where $m^2 \equiv 1$, mod 6. The only solutions are $m = 1$, or 5. The latter gives a dihedral group and the map lies on a sphere. Hence, S and T are permutable and ST is of order 6 which is the corresponding value of v in the table above. Therefore, *there exists a map of two hexagons on a surface of genus two. The map is self-dual.*

If $n = 8$ we have $TST = S^m$ where $m^2 \equiv 1$, mod 8. The solutions are $m = 1$, 3, 5, and 7. The last gives the dihedral group of order 16 and need not be considered. The first gives ST of order 8 which is not the proper value for v. For $m = 3$ we may take $S = (abcdefgh)$ and $T = (bd)(cg)(fh)$, whence $ST = (adeh)(bgfc)$ which is of the proper order. Hence, *there exists a map of two octagons on a surface of genus two.* If $m = 5$ we may take S to be as above in which case T will be $(bf)(dh)$; their product is $(afgdebch)$ which is not of the proper order. Hence, *there is just one map of two octagons.*

Finally, if $n = 12$ $TST = S^m$ where $m^2 \equiv 1$, mod 12. The possible solutions are $m = 1$, 5, 7, 11. The first and last are impossible as in the preceding case. For the other two cases we may take S to be $(abcdefghijkl)$ and T and T' to be $(bf)(ck)(ei)(hl)$ and $(bh)(dj)(fl)$ respectively. The orders of ST and ST' are 4 and 6 and therefore the groups give maps on surfaces of genera 3 and 4 respectively. Hence, *there exists no map of two dodecagons on a surface of genus two.*

We have now considered all the possibilities when k, the number of regions, is less than 3. We tabulate the remaining possibilities in the following list.

In making out this list we may proceed according to values of k as we have done for the cases $k = 1$ and 2. To see that k cannot be greater than 28 we may put the equation given by the Euler formula in the form $2/v + 2/n + 4/kn = 1$. Since $k > 28$ and $n \geq 3$, $1/v + 1/n \geqq 10/21$. Neither v nor n can be less than 3 and hence both must be less than 7, and if one of them is 4 the other must be less than 5. Possible pairs of values for v, n are 3, 3; 3, 4; 3, 5; 3, 6; 4, 3; 4, 4; 5, 3; 6, 3. The corresponding values of $n/2 - (n+v)/v$ are $-1/2$, $-1/4$, $-1/10$, 0, $-1/3$, 0, $-1/3$, 0. None of these values are positive as they must be to satisfy the Euler formula, since k is necessarily positive. Hence *there is no map of more than* 28 *regions on a surface of genus two.*

The columns headed r and g give respectively the number of vertices and the orders of the groups. The list contains just one of a pair of dual maps, the one given being the one with the smaller value of k, for example, the first map would contain 3 10-sided regions coming together 3 at a vertex and its dual which is not listed would be made up of 10 triangles coming together 10 at a vertex. The only exceptions we have made are in the cases where the dual map would contain 1 or 2 regions.

	k	n	v	r	g
1.	3	10	3	10	30
2.	3	4	12	1	12
3.	4	9	3	12	36
4.	4	6	4	6	24
5.	4	5	5	4	20
6.	4	4	8	2	16
7.	6	8	3	16	48
8.	6	3	18	1	18
9.	8	5	4	10	40
10.	8	3	12	2	24
11.	12	7	3	28	84.

We may dispose immediately of nos. 2, 6, 8, and 10. Since there does not exist a map of one 12-sided region or one 18-sided region on a surface of genus two *there exists no map corresponding to no.* 2 *or no.* 8. There exists a map of two octagons and no map of two dodecagons, hence *there exists a map of* 4 *4-sided regions* and *there does not exist a map of* 8 *triangles on a surface of genus two.*

A brief consideration shows that there can exist no regular map of 4 pentagons corresponding to no. 5. No region can touch itself along an edge unless it touches itself along every edge, in which case the map would consist of a single region of an even number of sides. If a given region of a regular map touches another region more than once along an edge it must touch each of the regions on its boundary the same number of times and hence if the number of its edges is a prime it can touch but one other region. Such a map can contain but two regions. Hence *there exists no map of* 4 *pentagons on a surface of genus two.*

For the remaining possibilities we examine the substitution groups of the proper order and degree, all of which, with one exception, have been listed.*

* These groups are to be found as follows:
Degrees 4–8, Miller, *American Journal of Mathematics*, Vol. 28 (1899), pp. 287-338;
Degree 10, Cole, *Quarterly Journal*, Vol. 27 (1894), pp. 39-50.
Degree 12, Miller, *Quarterly Journal*, Vol. 28 (1896), pp. 193-231.

The degree of the group may be taken to be either k or r, since r is the number of regions of the dual map. The exception is that of a possible group of order 48 and degree 6 or 16 corresponding to no. 7 of the list. It is evident that if such a map exists its group cannot be represented on symbols for regions, but may be represented on symbols for regions of the dual map. Thus we would seek a substitution group of order 48 and degree 16. The groups of degree 16 have not been listed.

We take up the remaining cases in order.

1. We seek a group of order 30 and degree 10. No such group exists and hence *there exists no map.*

3. There are five groups of order 36 and degree 12.

The first two groups contain $(abcdef\cdot ghijkl)_{18}$ which is simply isomorphic with $(\alpha\beta\gamma)$ all $(\delta\epsilon\zeta)$ cyc. G_{18} thus contains operators of orders 2, 3, and 6. The two groups are obtained by combining G_{18} with $T_1 = (ag\cdot bh\cdot ci\cdot dj\cdot ek\cdot fl)$ and $T_2 = (aj\cdot bl\cdot ck\cdot dg\cdot ei\cdot fh)$ respectively. T_1 is permutable with every operator of G_{18}, but T_2 is permutable with G_{18} without being permutable with every operator. Neither of these groups contains an operator of order 9, for, since $T_1ST_1 = S$ and $T_2ST_2 = S'$ for every S where S and S' are in G_{18}, $(T_1S)^9 = 1$ would imply $T_1S^9 = 1$ and $(T_2S)^9 = 1$ would imply $T_2S(S'S)^4 = 1$ which are impossible since neither T_1 nor T_2 is in G_{18}.

The third and fourth groups may be managed in the same manner. The third is $\{(abcdef)$ cyc $(ghijkl)$ cyc$\}$ pos and $R_1 = (ajdg\cdot bkeh\cdot clfi)$ whose square is in G_{18}. The fourth is $\{(abcdef)_6\ (ghijkl)_6\}$ pos and $R_2 = (ajdg\cdot blfh\cdot ckei)$ whose square is also in the corresponding G_{18}. The first G_{18} contains operators of orders 2, 3, and 6; the second contains operators of orders 2 and 3 since $(abcdef)_6$ is simply isomorphic with the symmetric group of degree 3 and the products of the type $(ad)(bf)(ce)(ghi)(jkl)$ are not in G_{18}. An operator of order 9 would have to be outside G_{18} in either case and hence of the form RS where R is either R_1 or R_2 and S is in the corresponding G_{18}. Since $RSR = R^4\cdot RSR = R^2R^3SR = R^2S'' = S'$ where S'' and S' are in G_{18}, $(RS)^9 = 1$ would imply $RS(S'S)^4 = 1$ which is impossible since R is not in G_{18}.

The fifth group is $(abcd\cdot efgh\cdot ijkl)$ pos which is simply isomorphic with the alternating group of degree 4, and $(aei\cdot bfj\cdot cgk\cdot dhl)$ which is permutable with every operator of the G_{12}. G_{12} contains operators of orders 2 and 3 and G_{36} contains operators of orders 2, 3, and 6. Hence, *there exists no map of twelve triangles on a surface of genus two.*

4. We seek a group of order 24 and degree 6. There are three transitive groups of the proper order and degree. The first two $(\pm\, abcdef)_{24}$ and $(+\, abcdef)_{24}$, are simply isomorphic with the symmetric group of degree 4 and so contain no operators of order 6. Hence, neither of these can be the group of a regular map of six 4-sided regions coming together six at a vertex.

The third group is $(abcdef)_{24_5}$ which is simply isomorphic with the direct product of the alternating group on four letters and an operator of order two. Its operators are of orders 2, 3, and 6 and so it could not be the group of a map of 4-sided regions.

7. We seek a map of 16 triangles coming together 8 at a vertex. The dual map is composed of six octagons coming together three at a vertex. Since in the dual map $n > k$ each octagon would have to touch just four or two others and it would be impossible to represent the group on symbols for regions. Therefore we have no hope of finding the group among those of order 48 and degree 6. The groups of order 48 and degree 16 have not been listed. However, the group generated by S and T which satisfy the relations

$$S^8 = 1, \quad T^2 = 1, \quad (ST)^3 = 1, \quad \text{and} \quad (S^4T)^2 = 1$$

is of order 48.* To see this we note that S,

$$S_1 = TST, \quad S_2 = S^{-1}S_1S = S^6T, \quad S_3 = S^{-1}S_2S = S^5TS,$$
$$S_4 = S^{-1}S_3S = TS^6, \quad \text{and} \quad S_5 = TS_3T = S^3$$

constitute a complete conjugate set, S transforming S_1, S_2, S_3, S_4 cyclically and leaving S_5 fixed and T transforming S, S_2, and S_3 into S_1, S_4, and S_5 respectively. S represented on symbols for its conjugates is of order four, T is of order two, and ST is of order three transforming S, S_1, and S_4 cyclically and S_2, S_5, and S_3 likewise cyclically. The group represented on symbols for conjugates of S is the group of the cube, or the octahedral group, being generated by two operators of orders two and three whose product is of order four. Hence, the group generated by S and T is of order 48. Therefore, *there exists a map of six octagons coming together three at a vertex on a surface of genus two.*

9. We seek a group of order 40 and degree 8 or 10. There is one group of order 40 and degree 10, viz. $(abcde \cdot fghij)_{20}$ and $T = (af \cdot bg \cdot ch \cdot di \cdot ej)$ which is permutable with every element of the G_{20}. The G_{20} is simply isomorphic with the metacyclic group of degree 5. The elements of G_{20} are of

* This group is among the groups of genus two given by Burnside. It is also the group described at the end of § 2.

orders 2, 4, and 5 and the elements outside of G_{20} are of orders 2, 4, and 10. The elements of order 5 with identity constitute a cyclic subgroup which is invariant under G_{20} and so under G_{40}. Let S and Σ be elements of orders 4 and 5 respectively of G_{20}. G_{40} is generated by S and an element of order two outside of G_{20} which must be of the form $T\cdot\Sigma^{-k} S^2\Sigma^k = T'$, for the group $[S, T']$ contains $S^2T' = TS^2 \Sigma^{-k} S^2 \Sigma^k = T\cdot\Sigma^{-k}$, since $S^{-1} \Sigma S = \Sigma^2$. $T\cdot\Sigma^{-k}$ is of order 10 and its square is a power of Σ, hence Σ is in $[S, T']$. The order of ST' is not 5 since all the operators of order 5 are in G_{20}. Hence, the map corresponding to such a choice of generators would not lie on a surface of genus two.* The generators might be chosen to be $S' = ST$ and $T'' = \Sigma^{-k} S^2 \Sigma^k$ the first being any operator of order four outside of G_{20} and the second being in G_{20}. The above argument shows also that their product is not of order 5. Noting that there exists no group of order 40 and degree 8, we have the result that *there exists no regular map of ten 4-sided regions on a surface of genus two.*

11. This case is readily disposed of by observing that there exists no group of order 84 and degree 12.

Collecting our results we see that there exist just 8 regular maps on a surface of genus two. They are made up of

1 octagon, a self-dual map;
1 decagon and the dual map of
2 pentagons;
2 hexagons, a self-dual map;
2 octagons and the dual map of
4 quadrangles;
6 octagons and the dual map of
16 triangles.

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* For neither choice of generators could ST be of order two since G_{40} is not dihedral; ST must therefore be of order four and the map lies on an anchor ring. See *Regular Maps on an Anchor Ring*, p. 234.