A strong Ramanujan theorem and the Riemann Hypothesis

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The sum of divisors function $\sigma(n)$

The function $\sigma(n) = \sum_{d|n} d$ is the *sum of divisors* function.

$$\sigma(1) = 1$$

$$\sigma(2) = 1 + 2 = 3,\ \sigma(3) = 1 + 3 = 4, \sigma(4) = 1 + 2 + 4 = 7$$

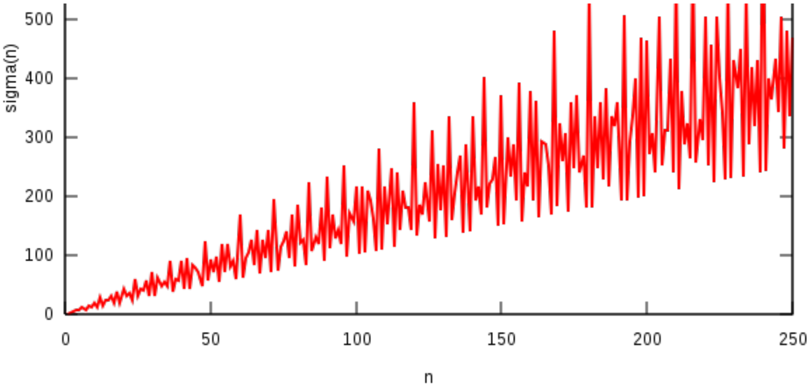
$$\sigma(6) = 1 + 2 + 3 + 6 = 12,\ \sigma(7) = 1 + 7 = 8,\ \sigma(8) = 15$$

1, 3, 4, 7, 6, 12, 8, 15, 13, 18, 12, 28, 14, 24, 24, 31, 18, 39, 20, 42, 32, 36, 24, 60, 31, 42, 40, 56, 30, 72, 32, 63, 48, 54, 48, 91, 38, 60, 56, 90, 42, 96, 44, 84, 78, 72, 48, 124, 57, 93, 72, 98, 54, 120, 72, 120, 80, 90, 60, 168, 62, 96, 104, 127, 84, 144, 68, 126, 96, 144

The sum of divisors function $\sigma(n)$

$\sigma(n) = n + 1$ if $n = 2, 3, 5, 7, 11, 13, 17, 19, \ldots$

$\sigma(n) = n + 1$ iff n is prime.



Euler's constant $\gamma \approx 0.5772$

Euler's (or Euler–Mascheroni's) constant
$\gamma = 0.57721566490153286060651209008240243104215933593 99\ldots$

$$\gamma := \lim_{n\to\infty} (H_n - \ln n), \quad H_n := 1 + 1/2 + \ldots + 1/n.$$

$$\gamma = -\int_0^\infty e^{-x} \ln x \, dx$$

Grönwall theorem (1913)

Theorem (Grönwall)

$$\limsup_{n\to\infty} \frac{\sigma(n)}{n \log\log n} = e^{\gamma}$$

$$G(n) := \frac{\sigma(n)}{n \log\log n}, \quad n \geq 2$$

$$\limsup_{n\to\infty} G(n) = e^{\gamma},$$

Robin theorem

Robin (1984) showed that the Riemann hypothesis (RH) is true iff

$$\sigma(n) < e^{\gamma} n \log \log n \quad \textit{for all} \quad n > 5040 \qquad (R)$$

or equivalently

$$G(n) < e^{\gamma} \; \forall n > 5040.$$

Briggs' computation of the colossally abundant numbers implies (R) for $n < 10^{(10^{10})}$.

According to Morrill and Platt (2018), (R) holds for all integers $5040 < n < 10^{(10^{13})}$.

Lagarias theorem

J. C. Lagarias. An Elementary Problem Equivalent to the Riemann Hypothesis. *Am. Math. Monthly*, 109 (2002), 534–543.

Theorem (Lagarias)

The RH is true iff

$$L(n) := H_n + \exp(H_n)\log(H_n) - \sigma(n) > 0 \quad \textit{for all} \quad n > 1. \qquad (L)$$

Recall

$$H_n := 1 + 1/2 + ... + 1/n.$$

SA and CA numbers

The study of numbers with $\sigma(n)$ large was initiated by Ramanujan.

A positive integer n is called *superabundant* (SA) if

$$\frac{\sigma(k)}{k} < \frac{\sigma(n)}{n} \quad \text{for all integer} \quad k \in [1, n-1].$$

Colossally abundant numbers (CA) are those numbers n for which there is $\varepsilon > 0$ such that

$$\frac{\sigma(k)}{k^{1+\varepsilon}} \leq \frac{\sigma(n)}{n^{1+\varepsilon}} \quad \text{for all} \quad k > 1.$$

CA numbers

$$F(x,k) := \frac{\log(1+1/(x+\ldots+x^k))}{\log x},$$

$$E_p := \{F(p,k) \mid k \geq 1\}, \quad p \text{ is a prime},$$

$$E := \bigcup_p E_p = \{\varepsilon_1, \varepsilon_2, \ldots\} = \left\{\log_2\left(\frac{3}{2}\right), \log_3\left(\frac{4}{3}\right), \log_2\left(\frac{7}{6}\right), \ldots\right\}.$$

CA numbers: Alaoglu–Erdős theorem

Alaoglu and Erdős (1944) showed that if ε is not *critical*, i.e. $\varepsilon \notin E$, then $\sigma(k)/k^{1+\varepsilon}$ has a unique maximum attained at the number n_ε. Moreover, if ε satisfies $\varepsilon_i > \varepsilon > \varepsilon_{i+1}$, $i = 1, 2, \ldots$, then n_ε is constant on the interval $(\varepsilon_{i+1}, \varepsilon_i)$.

$$n_\varepsilon = \prod_{p \in \mathbb{P}} p^{a_\varepsilon(p)}, \quad a_\varepsilon(p) = \left\lfloor \frac{\log(p^{1+\varepsilon} - 1) - \log(p^\varepsilon - 1)}{\log p} \right\rfloor - 1$$

The first 14 CA numbers $n_1, \ldots, n_{14}$ are $2, 6, 12, 60, 120, 360, 2520, 5040, 55440, 720720, 1441440, 4324320, 21621600, 367567200$.

Ramanujan inequalities

Ramanujan (1915, 1997) proved that if n is a CA number (he called CA numbers as *generalized superior highly composite*) then under the RH the following inequalities hold

$$\limsup_{n\to\infty}\left(\frac{\sigma(n)}{n}-e^{\gamma}\log\log n\right)\sqrt{\log n}\le -c_1, \tag{1}$$

$$c_1 := e^{\gamma}(2\sqrt{2}-4-\gamma+\log 4\pi)\approx 1.3932$$

$$\liminf_{n\to\infty}\left(\frac{\sigma(n)}{n}-e^{\gamma}\log\log n\right)\sqrt{\log n}\ge -c_2, \tag{2}$$

$$c_2 := e^{\gamma}(2\sqrt{2}+\gamma-\log 4\pi)\approx 1.5578.$$

Ramanujan's inequalities

$$T(n) := \left(e^{\gamma} \log\log n - \frac{\sigma(n)}{n} \right) \sqrt{\log n}.$$

It is easy to see that Ramanujan's inequalities (1) and (2) yield the following fact:

If the RH is true, then there is i_0 such that for all CA numbers n_i, $i \geq i_0$, we have

$$1.393 < T(n_i) < 1.558 \tag{3}$$

The Strong Ramanujan Theorem (SRT)

Note that (2) does not hold for all integers. If p_i is prime, then $\sigma(p_i) = p_i + 1$. Therefore, $\limsup_{i\to\infty} T(p_i) = \infty$.
However, (1) holds for all numbers.

Theorem (The Strong Ramanujan Theorem; M., 2019)

If the RH is true, then

$$\liminf_{n\to\infty} T(n) \geq c_1 > 1.393.$$

Open problem: *Can Ramanujan's constant c_1 be improved?*

Ramanujan Theorem

SRT implies the following inequality:

If the RH is true, then there is n_0 *such that for all* $n > n_0$ *we have*

$$\sigma(n) + \frac{1.393\, n}{\sqrt{\log n}} < e^{\gamma} n \log\log n \tag{4}$$

which is stronger than *Ramanujan's theorem*:

If the RH is true, then there is n_0 *such that for all* $n > n_0$ *we have*

$$\sigma(n) < e^{\gamma} n \log\log n. \tag{5}$$

Proof of the Strong Ramanujan Theorem

(1) For every non–CA $n > 1$ there is $i > 1$ such that $n_{i-1} < n < n_i$, where n_{i-1} and n_i are two consecutive CA numbers. Robin (1984) showed that $G(n) \leq \max(G(n_{i-1}), G(n_i))$.

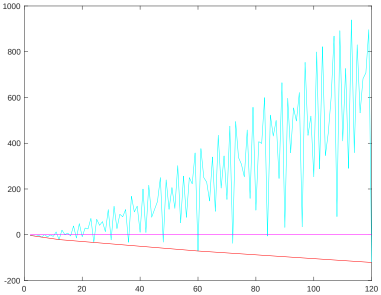
(2) Let $P(n)$ denote the largest prime factor of n. Alaoglu & Erdős proved that $P(n) \sim \log n$ for all SA (in particular for CA) numbers.

(3) The quotient of two consecutive CA numbers is either a prime or the product of two distinct primes [Alaoglu and Erdős].

Lower Convex Envelope

Let $D = \{x_n\}$ be an increasing sequence and $f : D \to \mathbb{R}$. Denote by $\Omega(f)$ the set of all convex functions $h : D \to \mathbb{R}$ such that $h(x) \leq f(x)$ for all $x \in D$. The *lower convex envelope* $\check{f}$ of f:

$$\check{f}(x) := \sup\{h(x) \mid h \in \Omega(f)\}.$$



Another definition of CA numbers

For fixed $\varepsilon > 0$, CA numbers n may be viewed as maximizers of

$$Q(k) - \varepsilon \log k = \log(\sigma(k)/k^{1+\varepsilon}), \quad Q(k) := \log \sigma(k) - \log k.$$

$$x_k := \log k, \quad A(x_k) := x_k - \log \sigma(k) = -Q(k),$$

$$A : D \to \mathbb{R}, \quad D := \{x_k\}, \quad k \geq 2$$

Note that n is CA if $(x_n, A(x_n))$ is a *vertex* of the lower convex envelope $\breve{A}$.

HA numbers

$$\mathrm{R_s(n)} := \left(\mathrm{e}^{\gamma}\mathrm{n}\log\log\mathrm{n} - \sigma(\mathrm{n})\right)(\log\mathrm{n})^{\mathrm{s}}, \quad \mathrm{n} \geq 2.$$

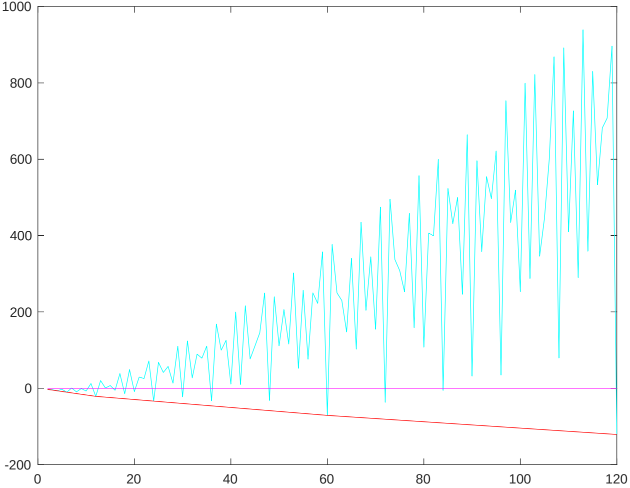
Now we define *Highest Abundant* (HA) numbers. We say that $n \in D \subset \mathbb{N}$ is HA with respect to $\mathrm{R_s}$ and write $n \in \mathrm{HA_s(D)}$ if for some real a

$$\mathrm{R_s(k) - ak}$$

attains its minimum on D at n. For $D = \{n \in \mathbb{N} \mid n \geq 5040\}$ we denote $\mathrm{HA_s(D)}$ by $\mathrm{HA_s}$.

Equivalently, $n \in \mathrm{HA_s(D)}$ if $(n, \mathrm{R_s}(n))$ is a vertex of the convex envelope $\breve{\mathrm{R}}_s$ on D.

The convex envelope of R_1 on $D = \{2, ..., 120\}$



$\mathrm{HA}_1(\mathrm{D})$ with $D = \{2, ..., n_{13} = 21621600\}$.

If $D = \{2, 3, ..., n_{13} = 21621600\}$, then

$\mathrm{HA}_1(\mathrm{D}) = \{2, 6, 12, 60, 120, 2520, 5040, 55440, 720720, 1441440,$ $2162160, 4324320, 21621600\} = \{m_0, ..., m_{12}\}$.

In this list of 13 numbers $m_0, ..., m_{12}$ there are 12 out of the first 13 CA numbers except $n_6 = 360$. However, m_{10} is an SA number $2162160 = 2^4 \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 13$ but is not CA.
R_1 on $\mathrm{HA}_1(\mathrm{D})$ has a minimum at $m_5 = 2520$ and is positive for $m_i > m_6 = 5040$.

Theorem 2

Theorem

(i) If the RH is true and $s > 1/2$, then there are infinitely many HA numbers with respect to R_s. If the RH is false, then $\mathrm{HA_s}$ is empty.

(ii) Let $s \leq 0$. If the RH is false, then there are infinitely many HA numbers with respect to R_s. If the RH is true, then $\mathrm{HA_s} = \{5040\}$.

Proof of Theorem 2

I. The SRT and Ramanujan inequality (2) yield

Corollary

If the RH is true, then for every $\varepsilon > 0$ *there is* n_0 *such that a set*

$$M(\varepsilon) := \{n > n_0 \mid T(n) < c_2 + \varepsilon\}$$

is infinite and for all $n \in M(\varepsilon)$ *we have* $T(n) > c_1 - \varepsilon$.

II. From Robin's result follow that if the RH is false there exist constants $b \in (0, 1/2)$ and $c > 0$ such that there are infinitely many $n \in \mathbb{N}$ with

$$-\frac{0.6482\, n}{\log\log n} < \mathrm{R}_0(\mathrm{n}) < -\frac{\mathrm{c\, n} \log\log \mathrm{n}}{(\log \mathrm{n})^{\mathrm{b}}}.$$

Open problems

Suppose that the RH is true.

(1) Can Ramanujan's constant c_1 be improved?

(2) Let $s = 1/2$. Is $\mathrm{HA_s}$ infinite?

(3) Let $\bar{c}_1$ be the optimal (Ramanujan's) constant. Let $W(n) := T(n) - \bar{c}_1$. Find $\tau(n)$ and constants b_1, b_2 such that

$$\liminf_{n\to\infty} W(n)\tau(n) \geq b_1$$

and there are infinitely many n with $W(n)\tau(n) \leq b_2$.

(!) Suppose that the RH is false. Improve Robin's inequalities.

Thank you