Spherical contact graphs: open problems

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Contact graphs

Let X be a finite subset of a metric space M. Here we consider $M = \mathbb{S}^2$. Denote

$$\psi(X) := \min_{x,y \in X} \{\mathrm{dist}(x, y)\}, \text{ where } x \neq y.$$

The *contact graph* $\mathrm{CG}(X)$ is the graph with vertices in X and edges (x, y), $x, y \in X$ such that

$$\mathrm{dist}(x, y) = \psi(X)$$

Shift of a single vertex

Let X be a finite set in M. Let $x \in X$ be a vertex of $\mathrm{CG}(X)$ We say that there exists a shift of x if x can be slightly shifted to x' such that $\mathrm{dist}(x', X \setminus \{x\}) > \mathrm{dist}(x, X \setminus \{x\})$.

Irreducible contact graph

We say that the graph $\mathrm{CG}(X)$ is *irreducible* [Schütte - van der Waerden, Fejes Tóth] if there are no shift of vertices.

Properties of spherical irreducible contact graphs

Theorem

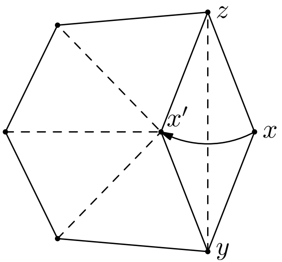
Let $X \subset \mathbb{S}^2$ with $|X| = N$ is such that the graph $\mathrm{CG}(X)$ is irreducible. Then $G := \mathrm{CG}(X)$ satisfies the following properties:

1. *G is a planar graph;*
2. *Any vertex of G is of degree $0, 3, 4,$ or 5;*
3. *If $N > 10$ and G contains an isolated vertex v, then v lies in a face with $m \geq 6$ vertices. Moreover, a hexagonal face of G cannot contain two or more isolated vertices.*

D–flip and D–irreducible contact graphs

Danzer [1963] defined the following flip. Let x, y, z be vertices of $\mathrm{CG}(X)$ with $\mathrm{dist}(x, y) = \mathrm{dist}(x, z) = \psi(X)$. We say that x is flipped over yz if x is replaced by its mirror image x' relative to the great circle yz. We say that this flip is *D (Danzer's)–flip* if $\mathrm{dist}(x', X \setminus \{x, y, z\}) > \psi(X)$.

If there are neither D–flips nor shifts of vertices, then we call $\mathrm{CG}(X)$ as a *D–irreducible graph.*



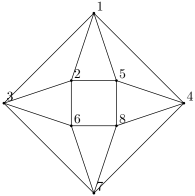
Danzer's work

In the Habilitationsschrift of Ludwig Danzer

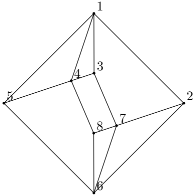
"Endliche Punktmengen auf der 2-sphäre mit möglichst großem Minimalabstand", Universität Göthingen, 1963,

are given all D-irreducible graphs for $6 \leq N \leq 10$.

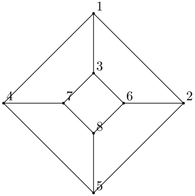
Danzer's work: D–irreducible contact graphs for $N = 8$



maximal $\mathcal{M}_8$



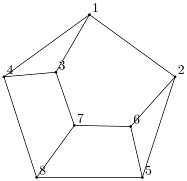
$\mathcal{M}_8(t)$

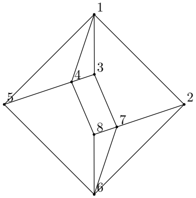


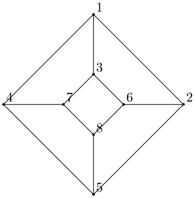
$\mathcal{M}_8(u, v)$

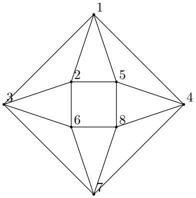
M. & Tarasov (2014): Irreducible graphs for N=8

N	d_{min}	d_{max}
1	1.17711	1.18349
2∗	1.28619	1.30653
3∗	1.23096	1.30653
4 ∗ ∗	1.30653	1.30653









$2n - 2$ conjecture

Conjecture. *The contact graph of an optimal spherical n–point configuration has at least $2n - 2$ edges.*

The Tammes problem

How must N congruent non-overlapping spherical caps be packed on the surface of a unit sphere so that the angular diameter of spherical caps will be as great as possible

Tammes PML (1930). "On the origin of number and arrangement of the places of exit on pollen grains". Diss. Groningen.

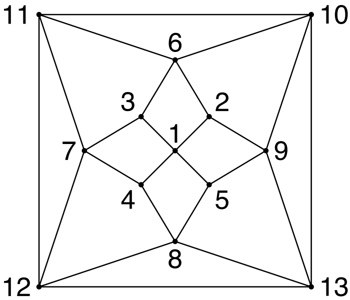
Maximal graphs G_N

Let X be a subset of $\mathbb{S}^2$ with $|X| = N$. We say that $\mathrm{CG}(X)$ is *maximal* if $\psi(X) = d_N$ and its number of edges is minimum. We denote this graph by G_N.
Actually, this definition does not assume that G_N is unique. We use this designation for some $\mathrm{CG}(X)$ with $\psi(X) = d_N$.

Proposition. Let $\mathrm{CG}(X)$ be a maximal graph G_N. Then for $N \geq 6$ the graph $\mathrm{CG}(X)$ is irreducible.

Tammes' problem for $N = 13$

The contact graph $\Gamma_{13} :=$CG(P_{13}) with $\psi(P_{13}) \approx 57.1367°$



Tammes' problem for $N = 14$

Theorem (M. & A. Tarasov). The arrangement of 14 points P_{14} in $\mathbb{S}^2$ is the best possible and the maximal arrangement is unique up to isometry.

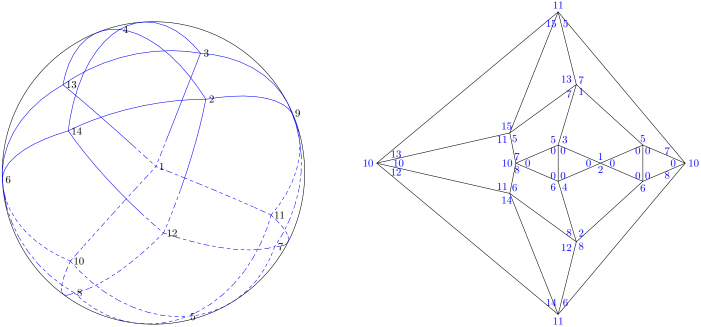


Figure: An arrangement of 14 points P_{14} and its contact graph Γ_{14} with $\psi(P_{14}) \approx 55.67057^\circ$.

Fejes Tóth's problem on maximum contacts

Let $X \subset \mathbb{S}^2$

$e(X) :=$ number of edges of the contact graph $\mathrm{CG}(X)$

$\mathrm{K}_N(d) :=$ max kissing (contact) number of N spherical caps with diameter d

$$\mathrm{K}_N := \max_{d \leq d_N} \mathrm{K}_N(d) = \max_{X \in \mathbb{S}^2, |X| = N} e(X)$$

Fejes Tóth's problem (1986): Find K_N.

Maximum contacts

$\kappa(d)$:=kissing number of the spherical cap with diameter d in $\mathbb{S}^2$.
If $d \leq \arccos(1/\sqrt{5})$, then $\kappa(d) = 5$.
We say that a packing of N spherical caps with diameter d is *maximal* if

$$\mathrm{K}_N(d) = N\,\kappa(d)/2.$$

Theorem (R. M. Robinson (1969), L. Fejes Tóth (1969))

A maximal packing of N equal spherical caps exists only if $N = 2, 3, 4, 6, 8, 9, 12, 24, 48, 60$ or 120.

$\mathrm{K}_8 = 16$, $\mathrm{K}_9 = 18$ and for $N \geq 12$ we have $\mathrm{K}_N = 5N/2$.

Maximum contacts

Theorem (M. & A. Tarasov, 2015)

1. $\mathrm{K}_5 = 8$ *(square pyramid);*
2. $\mathrm{K}_7 = 12$;
3. $\mathrm{K}_{10} = 21$;
4. $\mathrm{K}_{11} = 25$.

Open problem: *Find K_N for $N = 13, 14, ..., 23$.*

Maximum contacts: open problem

$$\mathrm{K}_{inf} := \lim_{N\to\infty} \inf \frac{\mathrm{K}_N}{N}, \quad \mathrm{K}_{sup} := \lim_{N\to\infty} \sup \frac{\mathrm{K}_N}{N}$$

We have

$$2 \leq \mathrm{K}_{inf} \leq \mathrm{K}_{sup} \leq 5/2$$

Open problem: *Find better bounds for* K_{inf} *and* K_{sup}.
Do we have the equality: $\mathrm{K}_{inf} = \mathrm{K}_{sup}$*?*

Uniqueness of maximal irreducible graphs

Problem. *Is it true that for $N > 5$ on $\mathbb{S}^2$ there is unique (up to isomorphism) maximal graph G_N?*

THANK YOU