Sphere packings in low dimensions

Oleg R. Musin

University of Texas Rio Grande Valley

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History: Thomas Harriot (1585)

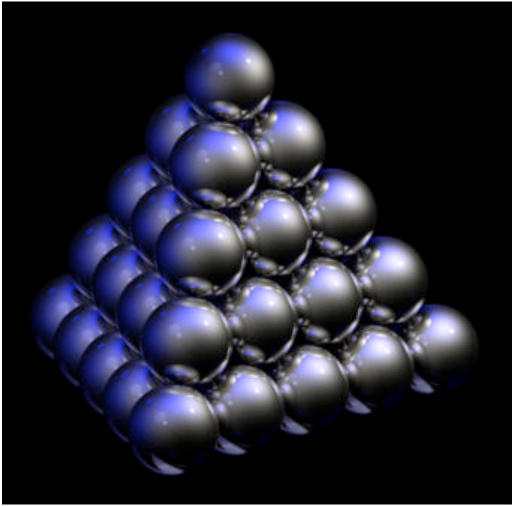
Sir Walter Raleigh's problem:

To develop a formula that would allow to know how many cannonballs can be in a given stack simply by looking at the shape of the pile.

Harriot discovered that for sufficiently large pile the highest density gives the so-called *face centered cubic* (FCC) packing. For the FCC packing the density is:

$$\frac{\pi}{3\sqrt{2}} \approx 0.74048$$

Face Centered Cubic (FCC) packing



History: Johannes Kepler (1611)

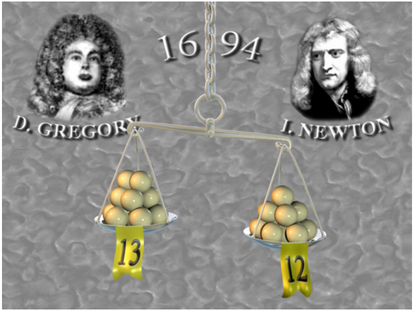
J. Kepler. The Six-Cornered Snowflake, 1611
In this little booklet Kepler examined several questions:
– Why honeycomb are formed as hexagon?
– Why the seeds of pomegranates are shaped as dodecahedra?
– Why the petals of flowers are most often grouped in fives?
– Why snowflakes are shaped as they are?

The Kepler Conjecture (1611):
The highest density of a packing of 3*-space by equal spheres* $= 0.74048\dots$

Hilbert's Problem 18:3 (1900):
"How can one arrange an infinite number of equal solids, of given form, most densely in space, e.g., spheres with given radii... How can one fit them together in a manner such that the ratio of the filled space to the unfilled space be as great as possible?"

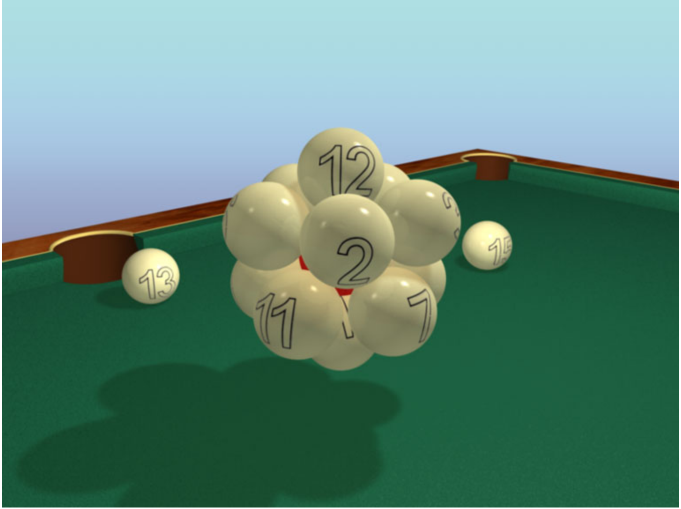
History: Gregory vs. Newton (1694)

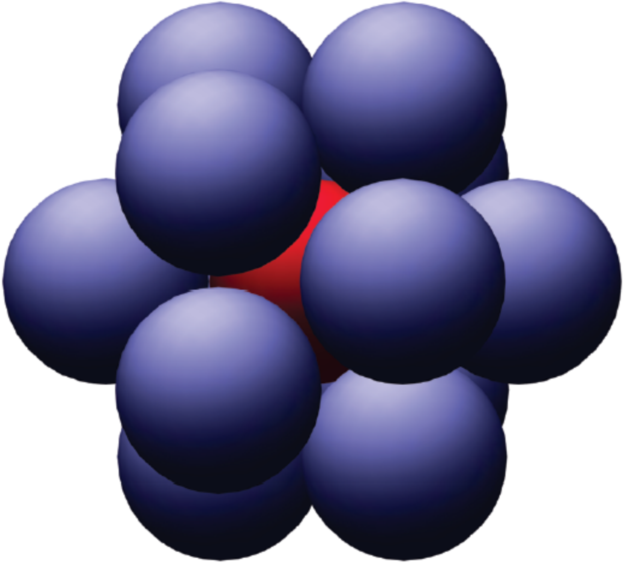
On May 4, 1694 David Gregory paid a visit to Cambridge for several days nonstop discussions about scientific matters with the leading scientist of the day Isaac Newton. Gregory making notices of everything that great master uttered. One of the points discussed, *number* 13, in Gregory's memorandum was 13 spheres problem.
Newton: $k(3) = 12$ vs.
Gregory: $k(3) = 13$ (The main Gregory argument was: area of the unit sphere $\approx 14.9\times$ area of a spherical cap of radius 30°.)



The Newton – Gregory problem = The thirteen spheres problem

The most symmetrical configuration, 12 billiard balls around another, is achieved if the 12 balls are placed at positions corresponding to the vertices of a regular icosahedron.





(Graphics: Detlev Stalling, ZIB Berlin)

History: Lattice sphere packing

Carl Friedrich Gauss (1831): *The FCC packing is the unique densest lattice sphere packing for dimension three.*

Hérmit (1850,1874); **Lebesgue** (1856); **Selling** (1874); **Minkowski** (1883), ..., **Mahler** (1992).

Korkine & Zolotareff: $n = 4$ (1872), $n = 5$ (1877).

Blichfeldt (1925, 1929, 1935): $n = 6, 7, 8$.

Cohn & Kumar (2009): $n = 24$.

History: Packing in the plane

Axel Thue provided the first proof that this was optimal in 1890, showing that

The hexagonal lattice is the densest of all possible circle packings, both regular and irregular.

However, his proof was considered by some to be incomplete. The first rigorous proof is attributed to László Fejes Tóth in 1940.

History: Hales (1998) and Viazovska (2016)

In 1998, **Thomas Callister Hales,** following the approach suggested by **László Fejes Tóth** in 1953, announced a proof of the Kepler conjecture. Hales' proof is a proof by exhaustion involving checking of many individual cases using complex computer calculations. On 10 August 2014 Hales announced the completion of a formal proof using automated proof checking, removing any doubt.

In 2016, **Maryna Viazovska** announced a proof that the E_8 lattice provides the optimal packing in eight-dimensional space, and soon afterwards she and a group of collaborators (**Cohn, Kumar, Miller, Radchenko**) announced a similar proof that the Leech lattice is optimal in 24 dimensions.

History: Schütte & v. d. Waerden (1953)

Reinhold Hoppe thought he had solved the thirteen spheres problem in 1874. However, there was a mistake — an analysis of this mistake was published by **Thomas Hales**: *The status of the Kepler conjecture*, Mathematical Intelligencer, 16 (1994), 47-58.
Finally, the thirteen spheres problem was solved by **Kurt Schütte** and **Baartel Leendert van der Waerden** in 1953. They had proved:

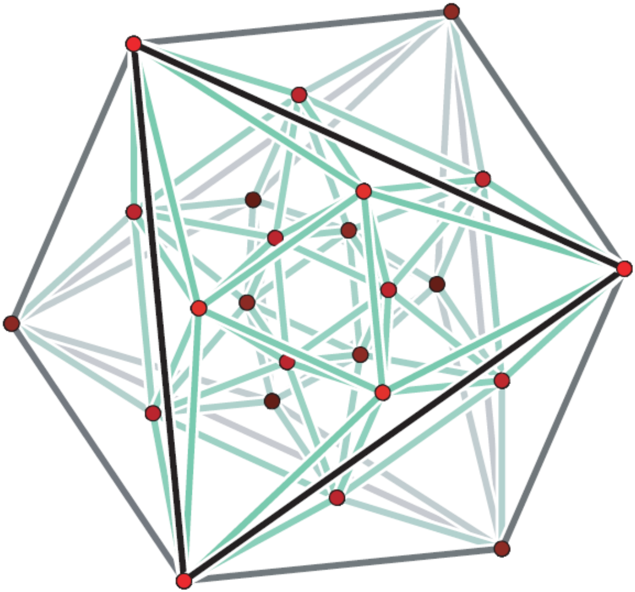
$$k(3) = 12.$$

It's not the end of the story about 13 spheres ...

John Leech(1956) : two-page sketch of a proof $k(3) = 12$.
...It also misses one of the old chapters, about the "problem of the thirteen spheres," whose turned out to need details that we couldn't complete in a way that would make it brief and elegant.
Proofs from THE BOOK, M. Aigner, G. Ziegler, 2^{nd} edition.
W. –Y. Hsiang (2001);
H. Maehara (2001, 2007);
K. Böröczky (2003);
K. Anstreicher (2004);
M. (2006)

k(4) is not less than 24

$n = 4$: There are 24 vectors with two zero components and two components equal to ± 1; they all have length $\sqrt{2}$ and a minimum distance of $\sqrt{2}$. Properly rescaled (that is, multiplied by $\sqrt{2}$), they yield the centers for a kissing configuration of unit spheres and imply that $k(4) \geq 24$. The convex hull of the 24 points yields a famous 4-dimensional regular polytope, the "24-cell", discovered in 1842 by Ludwig Schläfli. Its facets are 24 regular octahedra.



(Graphics: Michael Joswig/polymake [13])

The kissing problem in four dimensions

O.R. Musin, The kissing number in four dimensions // Annals of Math. 168 (2008), 1-32.

The proof relies on a combination of Delsarte's method and the irreducible contact graph method.

The kissing problem in four dimensions

$$f_4(t) = 53.76t^9 - 107.52t^7 + 70.56t^5 + 16.38t^4 - 9.83t^3 - 4.12t^2 + 0.434t - 0.016$$

Lemma

Let $P = \{v_1, \ldots, v_m\}$ be unit vectors in $\mathbb{R}^4$ (i.e. points on the unit sphere S^3). Then

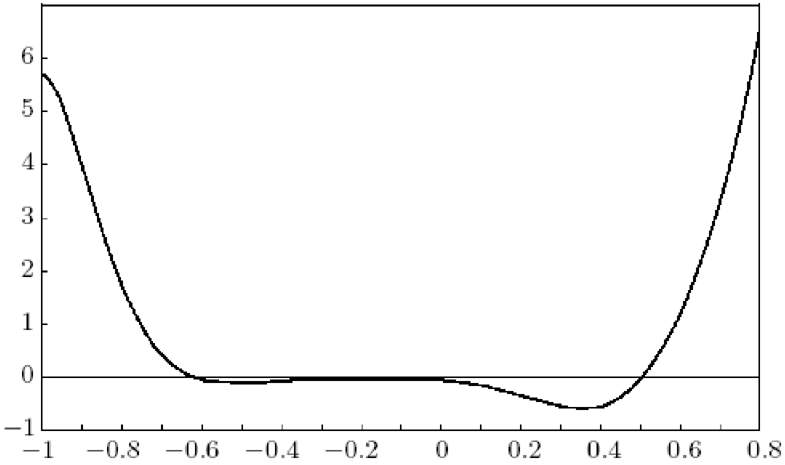
$$S(P) = \sum_{k,\ell} f_4(v_k \cdot v_\ell) \geq m^2.$$

Lemma

Let $P = \{v_1, \ldots, v_m\}$ be a kissing arrangement on the unit sphere S^3 (i.e. $v_k \cdot v_\ell \leq \frac{1}{2}$). Then

$$S(P) = \sum_{k,\ell} f_4(v_k \cdot v_\ell) < 25m.$$

The graph of $y = f_4(t)$



Theorem

$$k(4) = 24$$

Proof.

Suppose P is a kissing arrangement on S^3 with $m = k(4)$. Then P satisfies the assumptions in the lemmas. Therefore,

$$25m > S(P) \geq m^2$$

From this $m < 25$ follows, i.e. $m \leq 24$. From the other side: $m \geq 24$, showing that

$$m = k(4) = 24$$

□

Status 2019: Kissing numbers

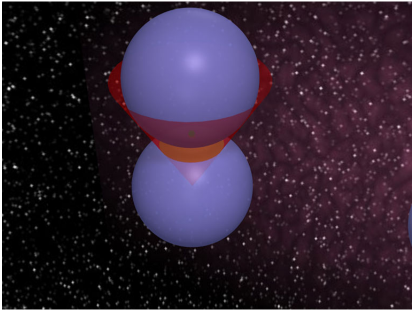
The only exact values of kissing numbers known:

n	lattice	regular polytope
$k(1) = 2$	A_1	
$k(2) = 6$	A_2	hexagon
$k(3) = 12$	H_3	icosahedron
$k(4) = 24$	$?D_4$	?24-cell
$k(8) = 240$	E_8	
$k(24) = 19650$	Λ_{24}	

In 1979: **V. I. Levenshtein** and independently **A. Odlyzko** and **N.J.A. Sloane** using Delsarte's method have proved that $k(8) = 240$, and $k(24) = 196560$.

Packing by spherical caps

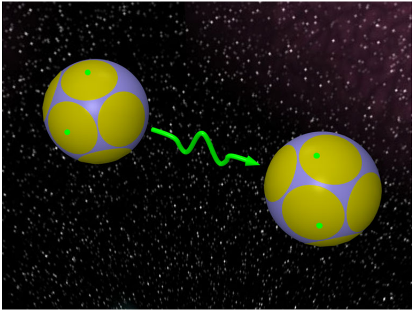
If unit spheres kiss the unit sphere S, then the set of kissing points is the arrangement on S such that the angular distance between any two points is at least 60°. Thus, the kissing number is the maximal number of nonoverlapping spherical caps of radius 30° on S.



Error–correcting spherical codes

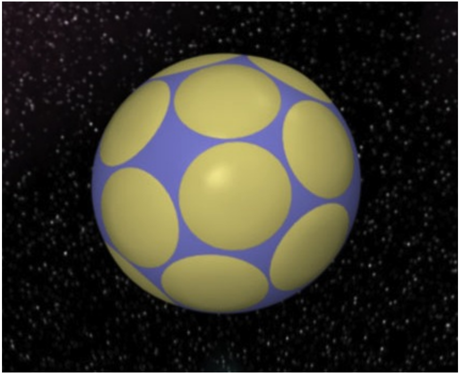
C. E. Shannon in [“A mathematical theory of communication”,1948] proposed to apply packings of the unit spheres by spherical caps of given radius r for *coding theory*.

The main application of this theory is in the design of signals for data transmission and storage.



Tammes' problem

Tammes' problem. How must N congruent non-overlapping spherical caps be packed on the surface of a unit sphere so that the angular diameter of spherical caps will be as great as possible



Tammes' problem

It is named after a Dutch botanist who posed the problem in 1930 while studying the distribution of pores on pollen grains. [Tammes P.M.L., *"On the origin of number and arrangement of the places of exit on pollen grains". * Diss. Groningen, 1930.]

This question is also known as the problem of the "inimical dictators":

Where should N dictators build their palaces on a planet so as to be as far away from each other as possible?

The Tammes problem

Let X be a finite subset of $\mathbb{S}^2$. Denote

$$\psi(X) := \min_{x,y \in X} \{\mathrm{dist}(x,y)\}, \text{ where } x \neq y.$$

Then X is a spherical $\psi(X)$-code.

Denote by d_N the *largest angular separation* $\psi(X)$ with $|X| = N$ that can be attained in $\mathbb{S}^2$, i.e.

$$d_N := \max_{X \subset \mathbb{S}^2} \{\psi(X)\}, \text{ where } |X| = N.$$

The Tammes problem

L. Fejes Tóth (1943): $N = 3, 4, 6, 12, \infty$

K. Schütte, and B. L. van der Waerden (1951): $N = 5, 7, 8, 9$

L. Danzer (1963): $N = 10, 11$

R. M. Robinson (1961): $N = 24$

M. & Tarasov: $N = 13$ and $N = 14$

N	d_N
4	109.4712206
5	90.0000000
6	90.0000000
7	77.8695421
8	74.8584922
9	70.5287794
10	66.1468220
11	63.4349488
12	63.4349488
13	57.1367031
14	55.6705700
…….	………………
15	53.6578501
16	52.2443957
17	51.0903285

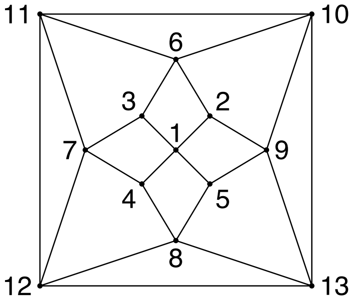
Contact graphs

The *contact graph* CG(X) is the graph with vertices in X and edges (x, y), $x, y \in X$ such that

$$\mathrm{dist}(x, y) = \psi(X)$$

N=13

The contact graph $\Gamma_{13} :=$CG(P_{13}) with $\psi(P_{13}) \approx 57.1367°$



Tammes' problem for $N = 13$

The value $d = \psi(P_{13})$ can be found analytically.

$$2\tan\left(\frac{3\pi}{8} - \frac{a}{4}\right) = \frac{1 - 2\cos a}{\cos^2 a}$$

$$d = \cos^{-1}\left(\frac{\cos a}{1 - \cos a}\right).$$

Thus, we have $a \approx 69.4051^\circ$ and $d \approx 57.1367^\circ$.

Tammes' problem for $N = 13$

Theorem (M. & A. Tarasov). The arrangement of 13 points P_{13} in $\mathbb{S}^2$ is the best possible, the maximal arrangement is unique up to isometry, and $d_{13} = \psi(P_{13})$.

Tammes' problem for $N = 14$

Theorem (M. & A. Tarasov) The arrangement of 14 points P_{14} in $\mathbb{S}^2$ is the best possible and the maximal arrangement is unique up to isometry.

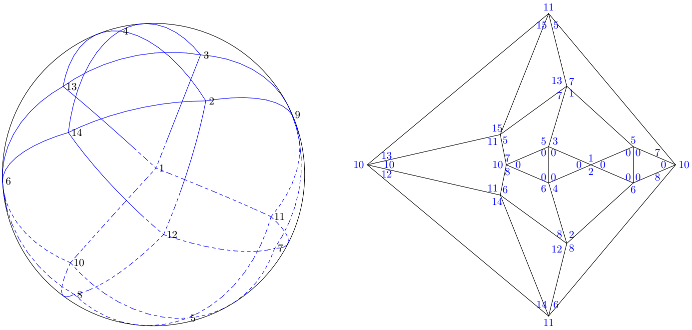


Figure: An arrangement of 14 points P_{14} and its contact graph Γ_{14} with $\psi(P_{14}) \approx 55.67057^\circ$.

Packing spheres by spheres: Methods

I. *Area inequalities.* L. Fejes Tóth (1943); for $d > 3$ Coxeter (1963) and Böröczky (1978)

II. *Distance and irreducible graphs.* Schütte, and van der Waerden (1951); Danzer (1963); Leech (1956);...

III. *LP and SDP.* Delsarte et al (1977); Kabatiansky and Levenshtein (1978); Odlyzko & Sloane (1978); Bachoc and Vallentin (2008); ...

Papers

O. R. Musin and A. S. Tarasov, *The strong thirteen spheres problem,* DCG, **48** (2012) 128–141.

O. R. Musin and A. S. Tarasov, *Enumeration of irreducible contact graphs on the sphere,* J. of Math Sciences, **203** (2014), 837–850

O. R. Musin and A. S. Tarasov, *Extreme problems of circle packings ...,* Proc. Steklv. Inst.,**288** (2015), 117–131

O. R. Musin and A. S. Tarasov, *The Tammes problem for N=14,* Experimental Math., **24:4** (2015), 460–468

O. R. Musin and A. V. Nikitenko, *Optimal packings of congruent circles on a square flat torus,* DCG, **55:1** (2016), 1–20.

Properties of the maximal contact graph G_N, $N = 13, 14$.

1. It is a planar graph with N vertices.
2. The degree of a vertex is 0,3,4, or 5.
3. All faces are polygons with m=3,4,5, or 6 vertices.
4. If there is an isolated vertex, then it lies in a hexagonal face.
5. No more than one vertex can lie in a hexagonal face.

Proof of Theorems

The proof consists of two parts:
(I) Create the list L_N of all graphs with N vertices that satisfy 1–5;
(II) Using linear approximations and linear programming remove from the list L_N all graphs that do not satisfy the geometric properties of G_N

The list L_{13}

To create L_{13} we use the program *plantri* (Gunnar Brinkmann and Brendan McKay). This program is the isomorph-free generator of planar graphs, including triangulations, quadrangulations, and convex polytopes. The program plantri generates 94,754,965 graphs in L_{13}. Namely, L_{13} contains 30,829,972 graphs with triangular and quadrilateral faces; 49,665,852 with at least one pentagonal face and with triangular and quadrilaterals; 13,489,261 with at least one hexagonal face which do not contain isolated vertices; 769,375 graphs with one isolated vertex, 505 with two isolated vertices, and no graphs with three or more isolated vertices.

Square packings

Obviously, the optimal packing in the torus could not be worse than the optimal packing in the unit square. Here are some results for the small number of disks in the square (Schaer & Meir (1965), Schaer (1965), and Melissen (1994)). Here d denotes the distance between the centers. Corresponding configurations are shown in Figure 1.

N	2	3	4	5	6	7	8	9
$d \approx$	0.586	0.509	0.500	0.414	0.375	0.349	0.341	0.333

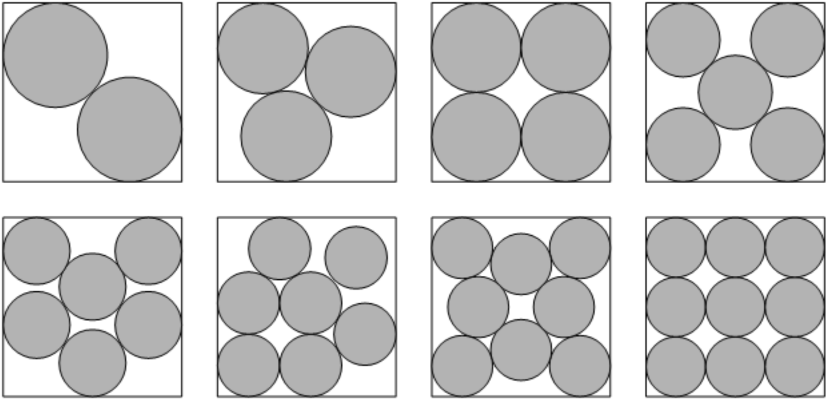


Figure: The optimal configurations for the square

The formal statement

The problem is: for a given number $N \geq 1$ of points, find the maximal $r \in \mathbb{R}^+$ such that N circles of radius r could be put on the square flat torus $\mathbb{T} = \mathbb{R}^2/\mathbb{Z}^2$ without overlapping, or, equivalently, to find the maximal $d \in \mathbb{R}^+$ such that there are N points on the torus with pairwise distances not less than d (where $d = 2r$).

$N = 7, 8$

Theorem (M. & A. Nikitenko). *There are three, up to isometry or up to a move of a free disk, optimal arrangements of 7 points in* $\mathbb{T}^2$ *which are shown in Figures 7.1 - 7.3 where* $d(7) = \frac{1}{1+\sqrt{3}} \approx 0.3660$.

Corollary (M. & A. Nikitenko). *There is one unique, up to isometry, optimal arrangement of 8 points in* $\mathbb{T}^2$*, which is shown in Figure 8 where* $d(8) = d(7) = \frac{1}{1+\sqrt{3}} \approx 0.3660$.

Optimal packings of circles on a square flat torus: 7.1

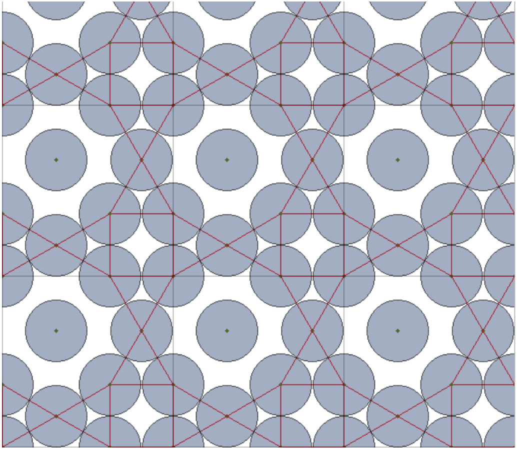


Figure: The first optimal configurations for N=7

Optimal packings of circles on a square flat torus: 7.2

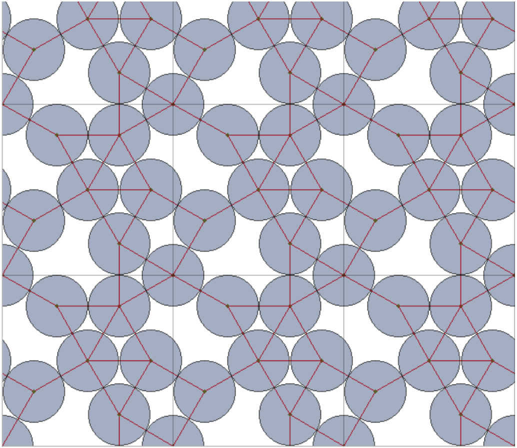


Figure: The second optimal configurations for N=7

Optimal packings of circles on a square flat torus: 7.3

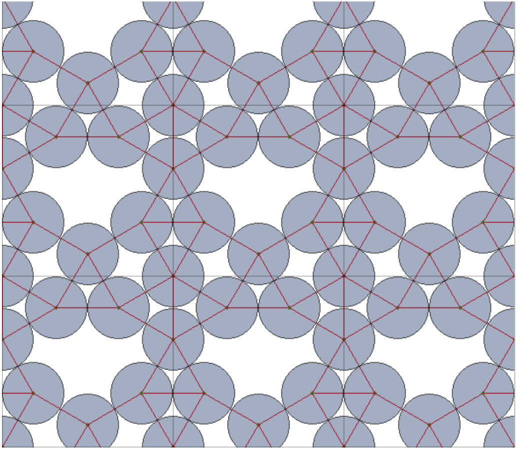


Figure: The third optimal configurations for N=7

Figure 8

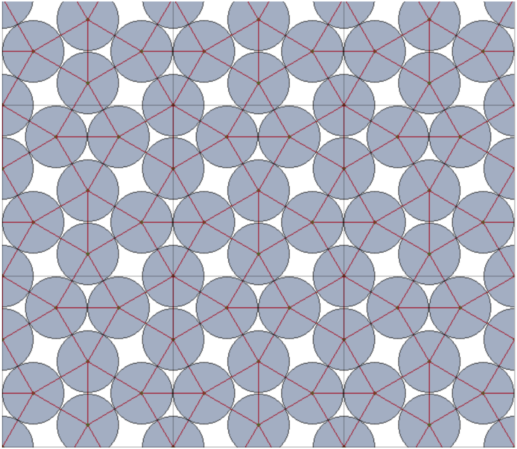


Figure: The optimal configuration for N=8

Furthure work: densest sphere packings in four dimensions

O. R. Musin, *Towards a proof of the 24–cell conjecture* // Acta Math Hungar., 155 (2018), 184–199

O. R. Musin, *An extension the semidefinite programming bound for spherical codes,* arXiv:1903.05767

The four dimensional lattice packing D_4

The checkerboard lattice $D_n := \{(x_1, \ldots, x_n) \in \mathbb{Z}^n : x_1 + \ldots + x_n \text{ even}\}$

$D_4^* = D_4$

The Voronoi cell of D_4 is the regular 24-cell

The density $\Delta_4 = \pi^2/16 = 0.6169...$

The densest packing by unit spheres in four dimensions is conjectured to be the D_4

The center density$=\Delta/B$:
$\mathrm{CD}_4 = 0.12500$;
Cohn–Elkies bound $= 0.13126$;
de Laat – de Oliveira Filho – Vallentin $= 0.130587$

The 24–cell conjecture

Consider the Voronoi decomposition of any given packing P of unit spheres in $\mathbb{R}^4$. The minimal volume of any cell in the resulting Voronoi decomposition of P is at least as large as the volume of a regular 24–cell circumscribed to a unit sphere.

dim=4: uniqueness of the maximal kissing arrangement

LP bound [Odlyzko & Sloane; Arestov & Babenko] = 25.558...

M. (2003): $k(4) < 24.865$

C. Bachoc & F. Vallentin (2008): $S_7(4) = 24.5797...$

H. D. Mittelmann & F. Vallentin (2010)
$S_{11}(4) = 24.10550859...$
$S_{12}(4) = 24.09098111...$
$S_{13}(4) = 24.07519774...$
$S_{14}(4) = 24.06628391...$

F.C. Machado & F.M. de Oliveira Filho (2018)
$S_{15}(4) = 24.062758...$
$S_{16}(4) = 24.056903...$

LP and SDP bounds

$$N \leq \frac{f(1)}{f_0}$$

$$N \leq \frac{f(1) + \hat{h}(n, T, f)}{f_0}$$

$$N^2 \leq \frac{F(1,1,1) + 3(N-1)B}{f_0}$$

$$N^2 \leq \frac{F(1,1,1) + 3(N-1)\,B + 3N\,\hat{h}(n, T, g)}{f_0}$$

Thank you