Structures and singularities in Image Processing

Discrete Analog of Maxwell-Morse Theory

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Related Publications and Projects:

Publications

O.R.Musin. Fast geometric transformation for Image Processing//International Journal of Imaging Systems and Technology, vol. 3, pp. 257-261, 1991.

O.R.Musin. On some problems of Computational Geometry and Topology// Lecture Notes in Mathematics, vol. 1520, pp.57-80, 1992.

O.R.Musin. Topographic Structure of Image// Lecture Notes in Computer Science, vol. 719, 1993.

V.A.Sadovnichy, V.V.Belokurov, M.I.Grinchuk, **O.R.Musin**, O.V.Seleznev, V.M. Staroverov.
Structures and Singularities in the Image Processing I. Moscow University press, 49 p., 1995

V.A.Sadovnichy, I.Antoniou, M.I.Grinchuk, **O.R.Musin**, O.V.Seleznev, V.M. Staroverov.
Structures and Singularities in the Image Processing II. Moscow University press, 56 p., 1996

V.A.Sadovnichy, I.Antoniou, **O.R.Musin**, O.V.Seleznev, V.M. Staroverov. Structural lines and singularities of the digital image // Selected papers…, MSU press, p. 438-462, 1999 (in Russian)

The Black Sea GIS. Edited by A.Berlyant, V.Mamaev, **O.Musin**, Astrea Press, 60 p., 1999.

O.R.Musin. Structural lines and digital elevation models//"The Interaction of Cartography and Geoinformatics", A.Berlyant and **O.Musin** (eds), Scientific World Press, pp. 21-34, 2000.

Projects

1995- 1999 **Solvay Institute** (Brussels) **& IMSCS** (Moscow) (CTIAC ESPRIT Program)

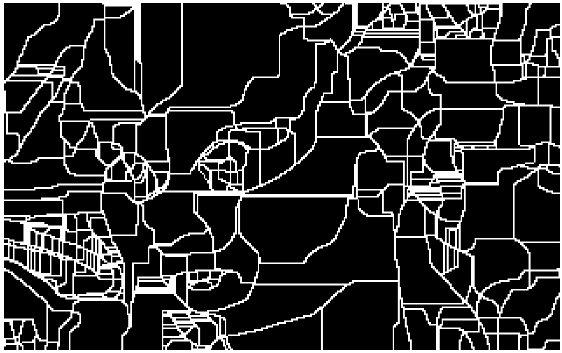
1994 – 2000 **Black Sea GIS (UNDP project)**

Black Sea Web (European Union project.)

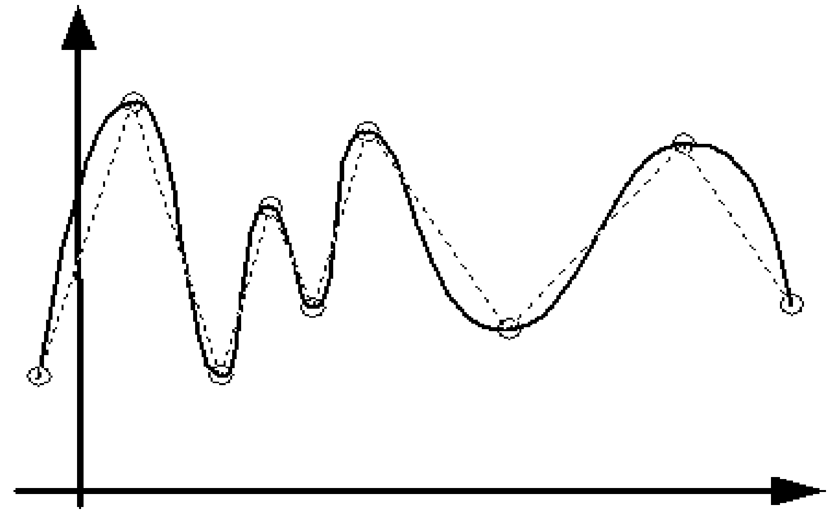
1997 – 1999 **Los Alamos National Lab (LANL)**

The main objectives

- Theory of critical points of discrete functions
- Discrete analog of gradient vector field
- Discrete analog of Maxwell-Morse theory
- Reconstruction of Discrete Function (=Digital Image or Digital Elevation Model) based on Structures
- Smoothing and enhancement of Discrete Function
- Compression of Discrete Function



In many applied and theoretical problems, the information about local maxima and minima of a signal (i.e., a function of one variable) is sufficient for its analysis. The distribution of extreme points defines the general behaviour of the function.



A generalisation of this approach related to an addition of information about singular points of the derivatives of higher order.

For the functions of two and more variables, singular points do not define its behaviour, since they do not define the subdivision of the function into areas without singular points.

2D-case: $z = f(x,y)$

- Critical (singular) points:

grad $f(x,y)=0$;

I – maxima

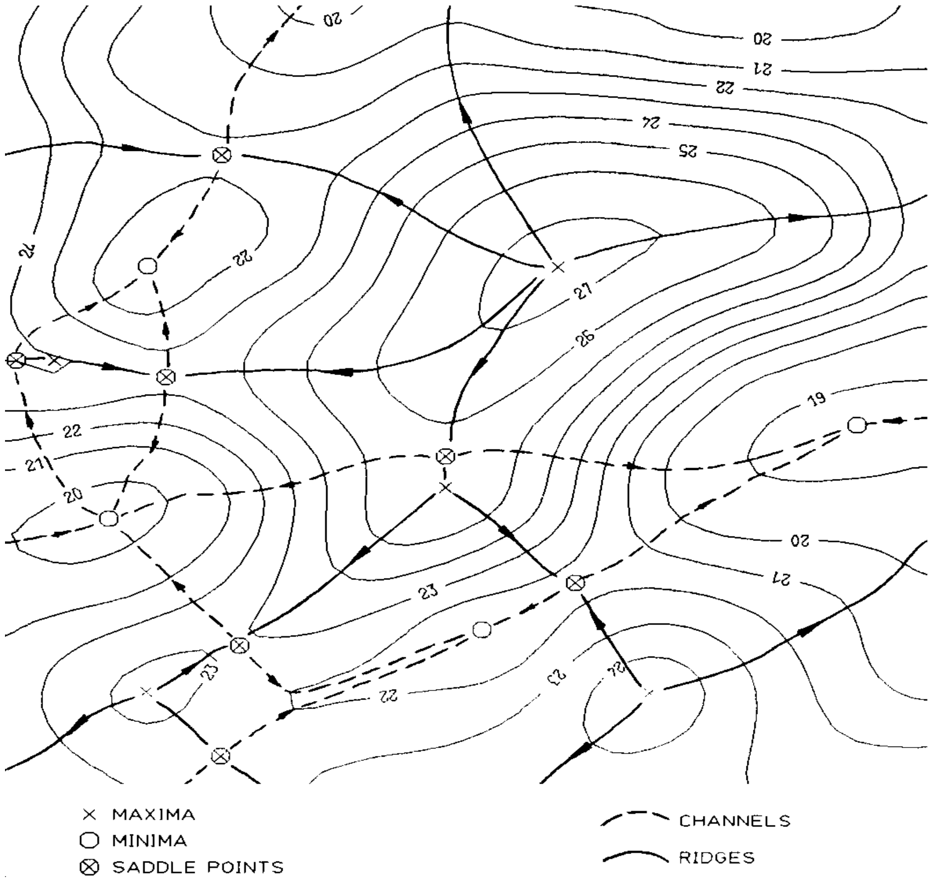
II – minima,

III – saddle points

- A.Cayley (1859)

I - summits, II - immits, III – knots

V = -grad (f) is gradient vector field



Separatrices

- The S is saddle point.

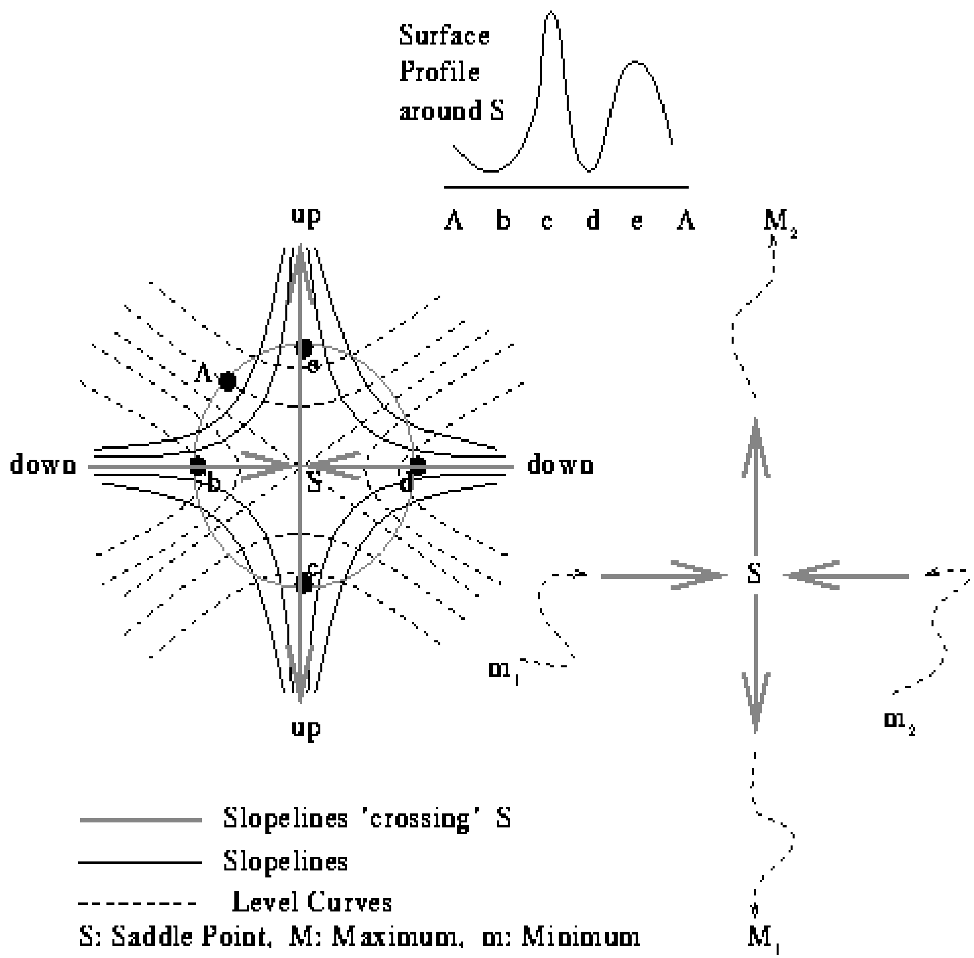
grad f(S)=0.

There are just two slopelines 'crossing' the S. Cayley termed these slopelines *ridge* and *course line*.

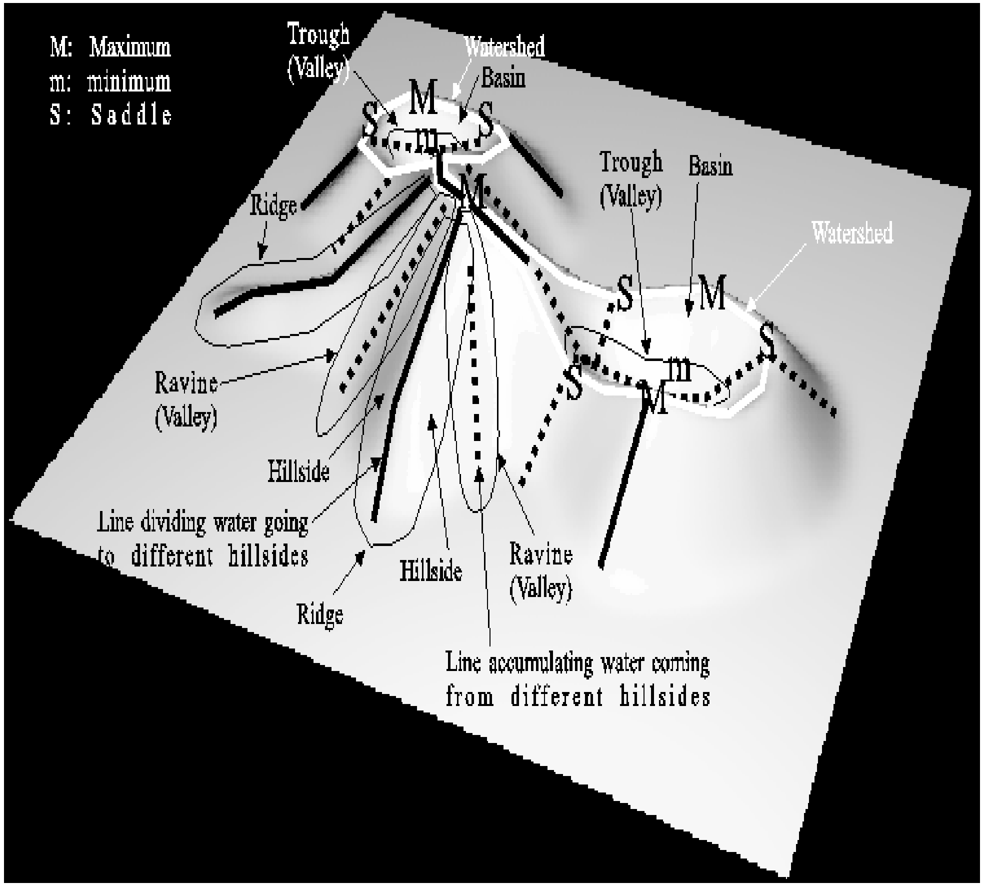
Course line = talweg (or thalweg) = watercouse = channel

Ridge = watershade

In Math separatrix is slopeline that going from local maximum to saddle or from saddle to local minimum = structural line (Geomorphology)



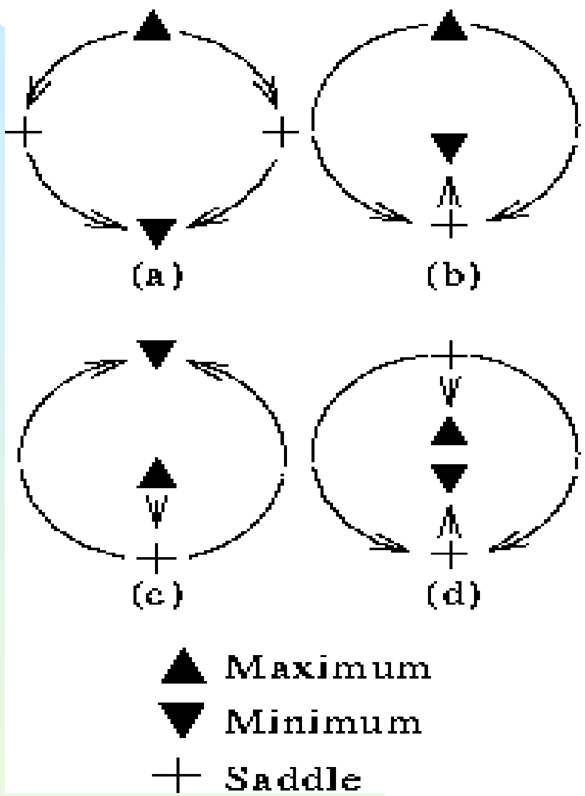
In 1870 J.C. Maxwell continued the work of Cayley. He defined the *basin* as district whose slope- lines run to the same min, and *hills* as districts whose slope-
lines come from the same max.



The hill H(f,p) is the basin B(-f,p), i.e. H(f,p) = B(-f,p).

The boundary of hills and basins are separatrices of the gradient vector field V(x,y)= -grad f(x,y).

Slope districts = Maxwell cells = Morse cells



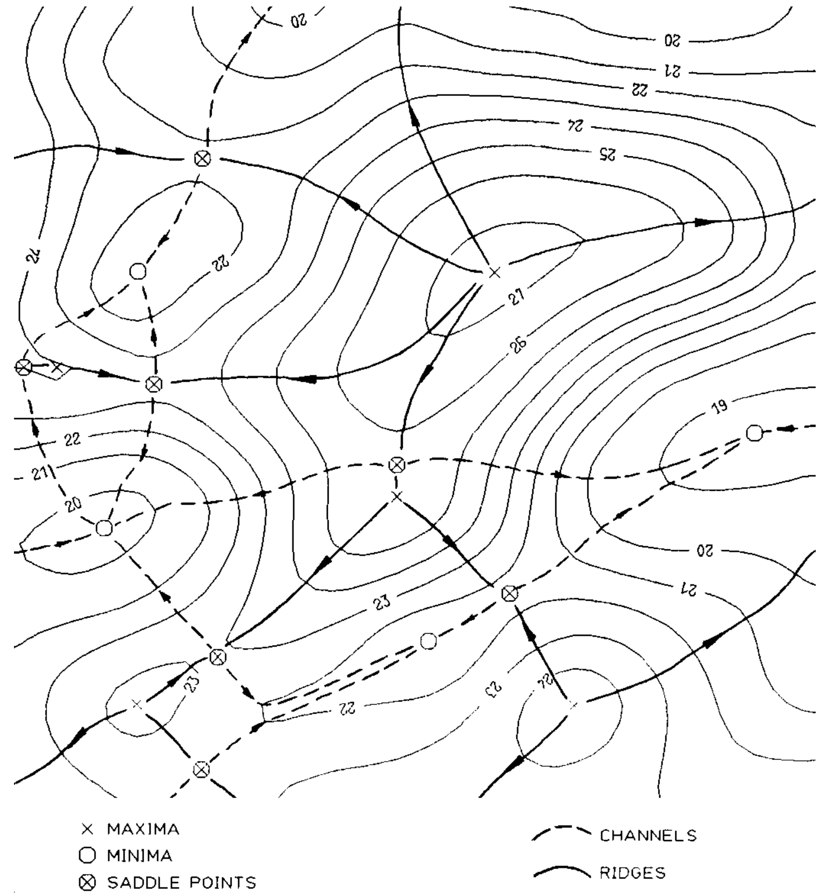
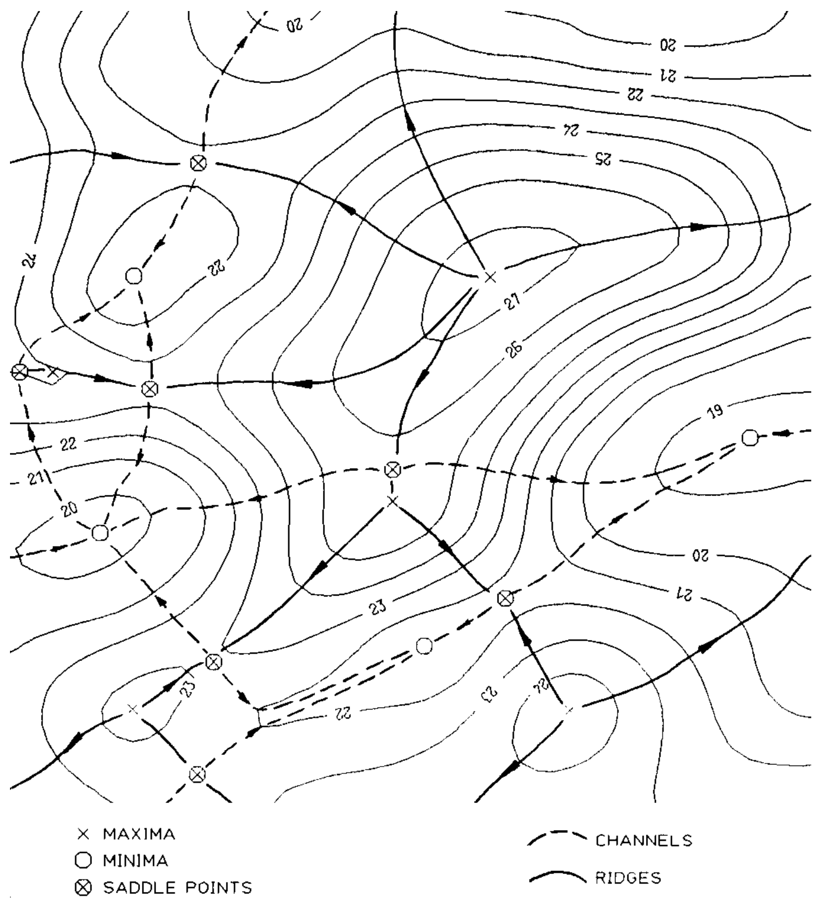


Fig. Canonical slope districts than can be constructed by following 'crossing' saddle points until a maximum/ minimum is reached

Our main purpose is to constrain discrete analogues of singularities (maximums, minimums, and saddle points) and structural lines for digital images.
Structure lines should includes the most essential pixels of an image.
The information, containing in the image structure points, has to be sufficient to reconstruct the initial image by means of this information with a given accuracy.



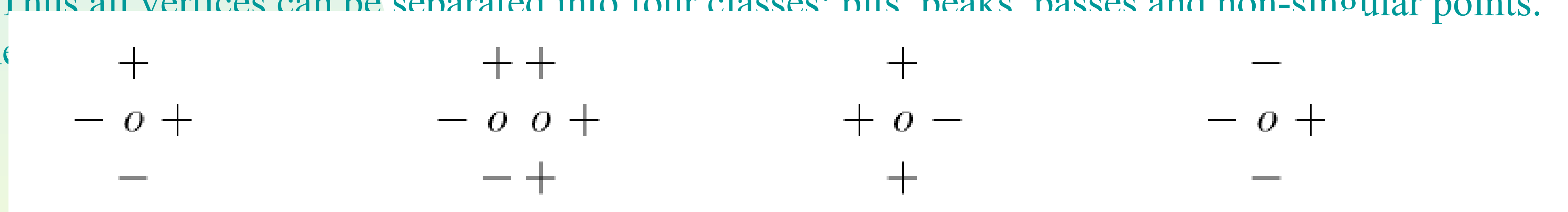
The main notion of our consideration is the ``flowing", i.e. the possibility of flow from one point to another (this notion is very similar to the notion ``gradient" in continuos case). We will give some definitions of this notion and will state some properties of the obtained relation map of image pixels. We will introduce also notions of ``channels" and ``watersheds".

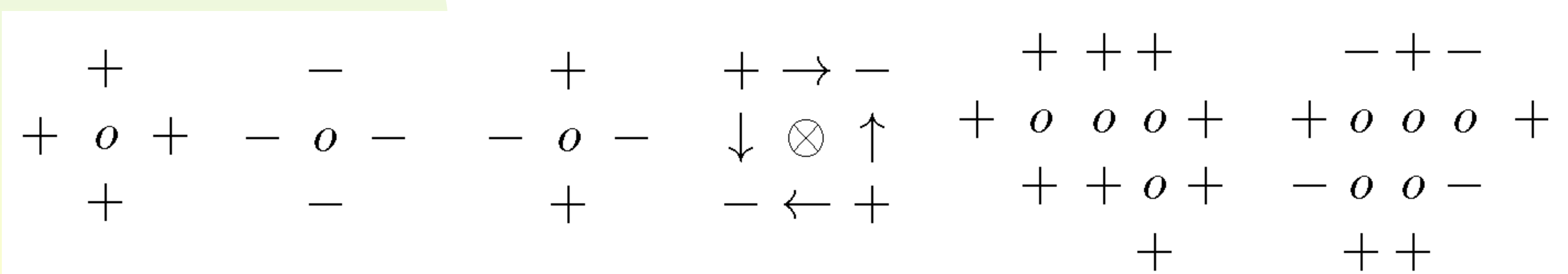
Critical points of discrete function (2D case)

To define *passes* (saddle points) and non-singular pixels, let us consider all of the vertices adjacent to the vertex v. We will mark the vertex u with the sign "+" if $f(v) > f(u)$ and with the sign "-" otherwise. It may be considered that vertices adjacent to v form a circle if v lies in the region, or are on an arc if v is on the boundary. Vertices on this circle (or arc) can be separated onto connected components with the signs '+' and '-'. We denote numbers of such components correspondingly by n_+ and n_- and we will name $1-n_-$ by *index* of the point, with the exception of a maximum point, which have index equals 1.

We can say that a vertex v is non-singular if $n_+ + n_- = 2$, and is the pass if $n_+ + n_- > 2$. We note that if the number of components is less than 2, then the vertex is either pit ($n_- = 0$) or peak($n_+=0$).

Thus all vertices can be separated into four classes: pits, peaks, passes and non-singular points. The





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To check a correctness of our definitions let us consider the well known theorem about the *index* of gradient vector field.

The index of vector field **V** in singular point *p* (i.e. **V**(*p*)=0) is equal winding number of **V** restricted to ``small'' circle around *p*.

The *index* of gradient vector field of function *f* is equal to 1 if the point is a maximum or minimum of *f*. The *index* is equal to 0 if the point is non-singular, and finally the *index* = -1, -2, -3, ..., for saddle points.

The sum of indices of singular points is a homotopy invariant, non depending on the function and equal to e(A) (Euler characteristic of the area A).

For a flat domain *e(A)* is equal *1-m,* where *m* is the number of the domain holes.

Let us consider following example.
In our example all points are extreme,
i.e. the index of all points is equal 1.
This example demonstrates that not only vertices but also cells can be singular.

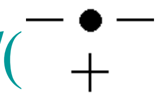
0	1	0	1	0
1	0	1	0	1
0	1	0	1	0
1	0	1	0	1
0	1	0	1	0

Index of point in discrete case

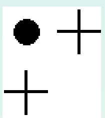
1) A point $p_{i,j}$ has 4 neighbours:

$ind(\max) = 1; \quad ind(\min) = 1; \quad ind(\text{saddle}) = -1$

2) A point $p_{i,j}$ has 3 neighbours:

ind() = 1; *ind*() = -1

3) A point $p_{i,j}$ has 2 neighbours:

ind()=1

4) If cell is a saddle then it has index = -1

The sum of indices of points on the rectangular grid is equal 1

(O.Musin 1992)

For a grid with *m* holes index is equal *1-m*

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Discrete analogue of gradient vector field

Definition 1. It is possible to flow from pixel ***(i,j)*** to pixel ***(k,l)*** if ***(k,l)*** is one of the neighbors of ***(i,j)*** and ***f(k,l)<=f(i,j)***.

Definition 2. It is possible to flow from pixel ***(i,j)*** to pixel ***(k,l)*** if ***(k,l)*** is one of the neighbors of ***(i,j)***, ***f(k,l) <= f(i,j)*** and ***f(i,j)-f(k,l)*** is maximum* with respect to ***(k,l)***.

Definition 3. It is possible to flow from pixel ***(i,j)*** to pixel ***(k,l)*** if ***(k,l)*** is one of the neighbors of ***(i,j)***, ***f(k,l) <= f(i,j)*** and ***f(i,j)-f(k,l)*** is maximum* with respect to ***(i,j)*** or ***(k,l)***.

Definitions 1 and 3 are reversible (if for a function ***f(i,j)*** it is possible fall from pixel ***(i,j)*** to pixel ***(k,l)***, than for function ***-f(i,j)***, it is possible to fall from the ***(k,l)*** pixel to pixel ***(i,j)***). Definition 2 is not reversible.

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\begin{array}{ccccccccccccc}
4 & \rightarrow & 3 & \rightarrow\leftarrow & 3 & \rightarrow & 2 & \leftarrow & 3 & \rightarrow\leftarrow & 3 & \leftarrow & 4 \\
\downarrow & & \uparrow & & \downarrow & & \downarrow & & \downarrow & & \uparrow & & \downarrow \\
3 & \leftarrow & 4 & \rightarrow & 2 & \rightarrow & 1 & \leftarrow & 2 & \leftarrow & 4 & \rightarrow & 3 \\
\uparrow\downarrow & & \downarrow & & \uparrow & & \uparrow & & \uparrow & & \downarrow & & \uparrow\downarrow \\
3 & \rightarrow & 1 & \leftarrow & 3 & \leftarrow & 4 & \rightarrow & 3 & \rightarrow & 1 & \leftarrow & 3 \\
\uparrow\downarrow & & \uparrow & & \downarrow & & \downarrow & & \downarrow & & \uparrow & & \uparrow\downarrow \\
3 & \leftarrow & 4 & \rightarrow & 2 & \rightarrow & 1 & \leftarrow & 2 & \leftarrow & 4 & \rightarrow & 3 \\
\uparrow & & \downarrow & & \uparrow & & \uparrow & & \uparrow & & \downarrow & & \uparrow \\
4 & \rightarrow & 3 & \rightarrow\leftarrow & 3 & \rightarrow & 2 & \leftarrow & 3 & \rightarrow\leftarrow & 3 & \leftarrow & 4
\end{array}
$$

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\begin{array}{ccccccccccccc}
* & \rightarrow & \odot & \rightarrow\leftarrow & \odot & \rightarrow & \odot & \leftarrow & \odot & \rightarrow\leftarrow & \odot & \leftarrow & * \\
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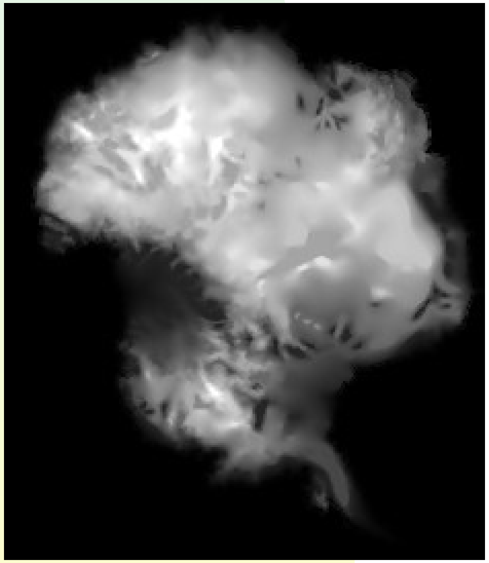
\end{array}
$$

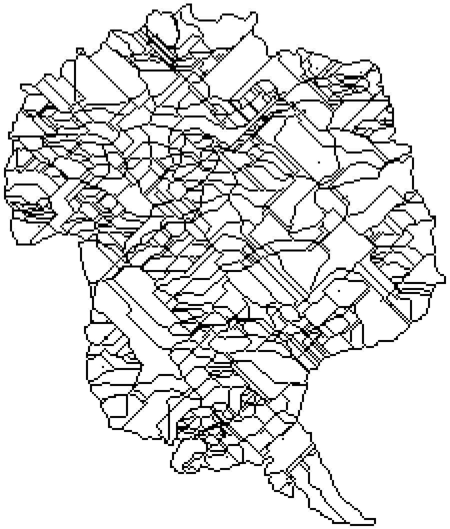
2D Discrete Analog of Maxwell-Morse Theory

We use three main approaches to definition "basin" and "hill" in the discrete case.

1. Structural lines (ridges and channels) are separatrices of a gradient vector field (J.C. Maxwell approach). Algorithm complexity in worth case is ***O(n)***.
2. Structure lines are boundaries of basins and hills. Algorithm complexity is ***O(n)***.
3. Ordering pixels by function value and sorting them by basins and hills. Algorithm complexity is ***O(n*** log ***n)***.

We proved the equivalence of these definitions for reversible definitions of gradient field.





Reconstruction of Images

We propose an algorithm of image reconstruction. It is admissible that we know values of a function for pixels of structure lines and for some specific points or lines (it is not important for the algorithm). The structure lines separate image in connected regions. Inside regions the function has no singularities. For reconstruction we use so-called *harmonic function*, i.e. a function the value of which at the vertex is equal to the average value of its neighbors.

For demonstration of the second method, we used the image of Black Sea digital elevation model (Image 1). Image 2 demonstrates the obtained structures lines. Image 3 is the image reconstructed from image structure lines.

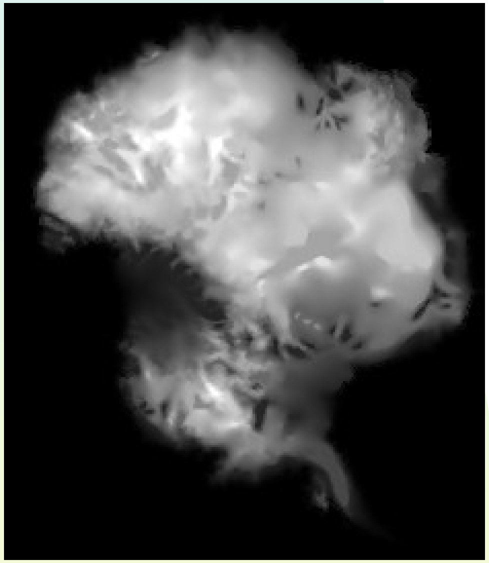


Image 1

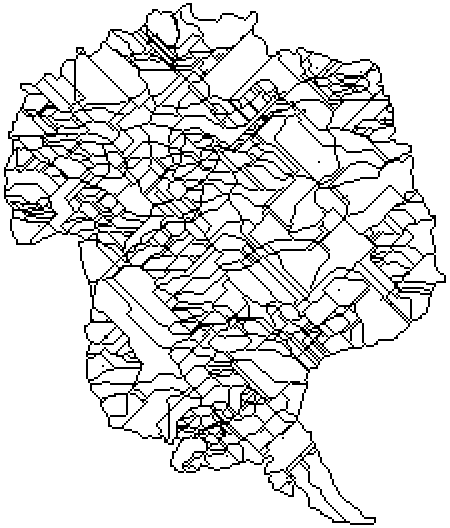


Image 2

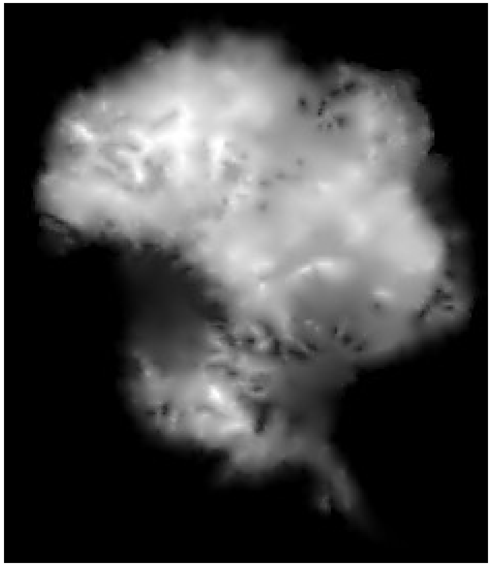
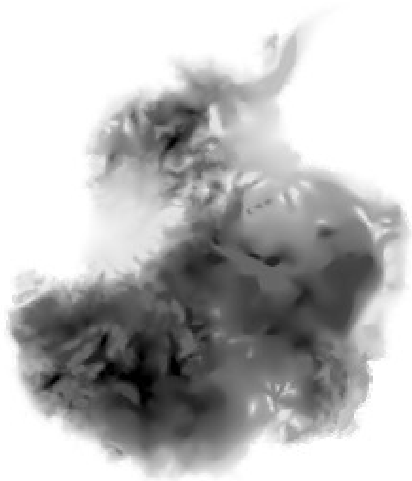
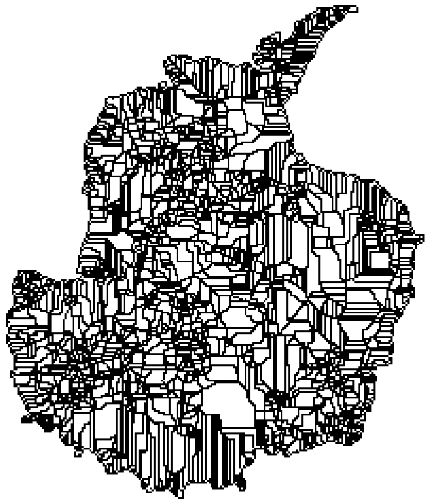


Image 3

Image Smoothing

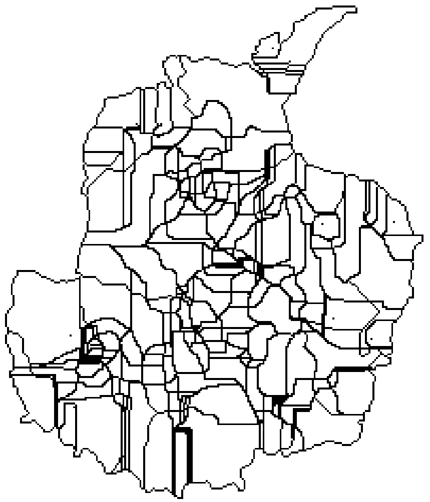


Black sea map



Structure lines of Black sea map
(#str.lines=33%; ε reconstr.=1.9%)

Smoothed Black sea map
(20 iteration by 1-4-1 filter)



Structure lines of Smoothed Black sea map
(#str.lines=10%; ε reconstr.=9.4%)

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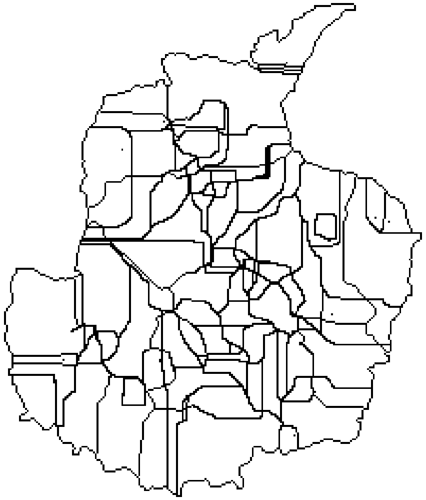
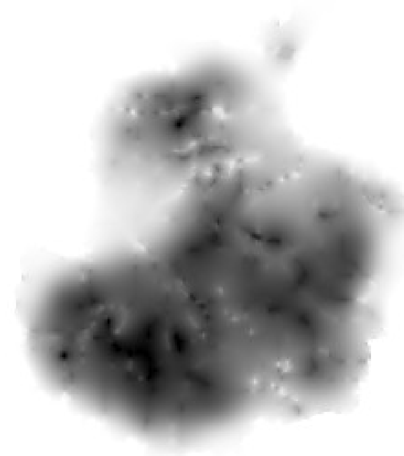
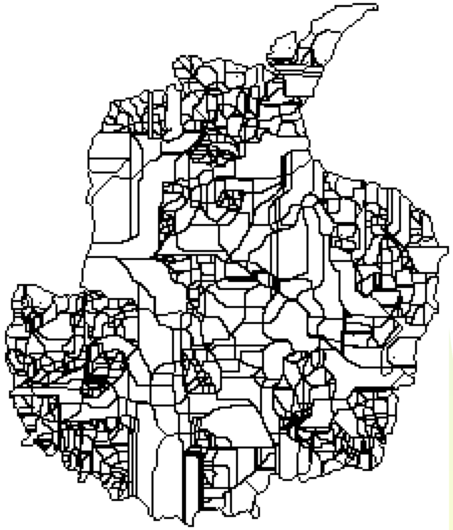
Generalized Structure lines



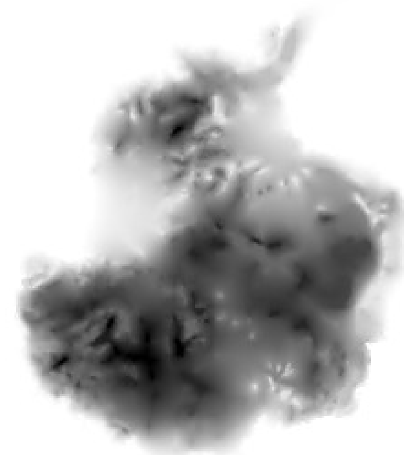
Generalized Str.lines of Black sea
(#str.lines=7.8%; ε reconstr.=12%)



Restored Black sea map



Generalized Str.lines of Black sea
(#str.lines=18%; ε reconstr.=4%)



Restored Black sea map

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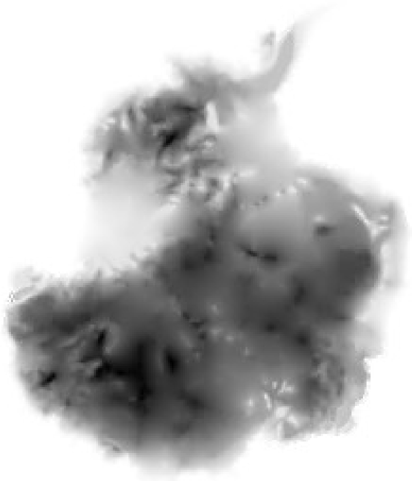
JPEG-like procedure of Image Compression

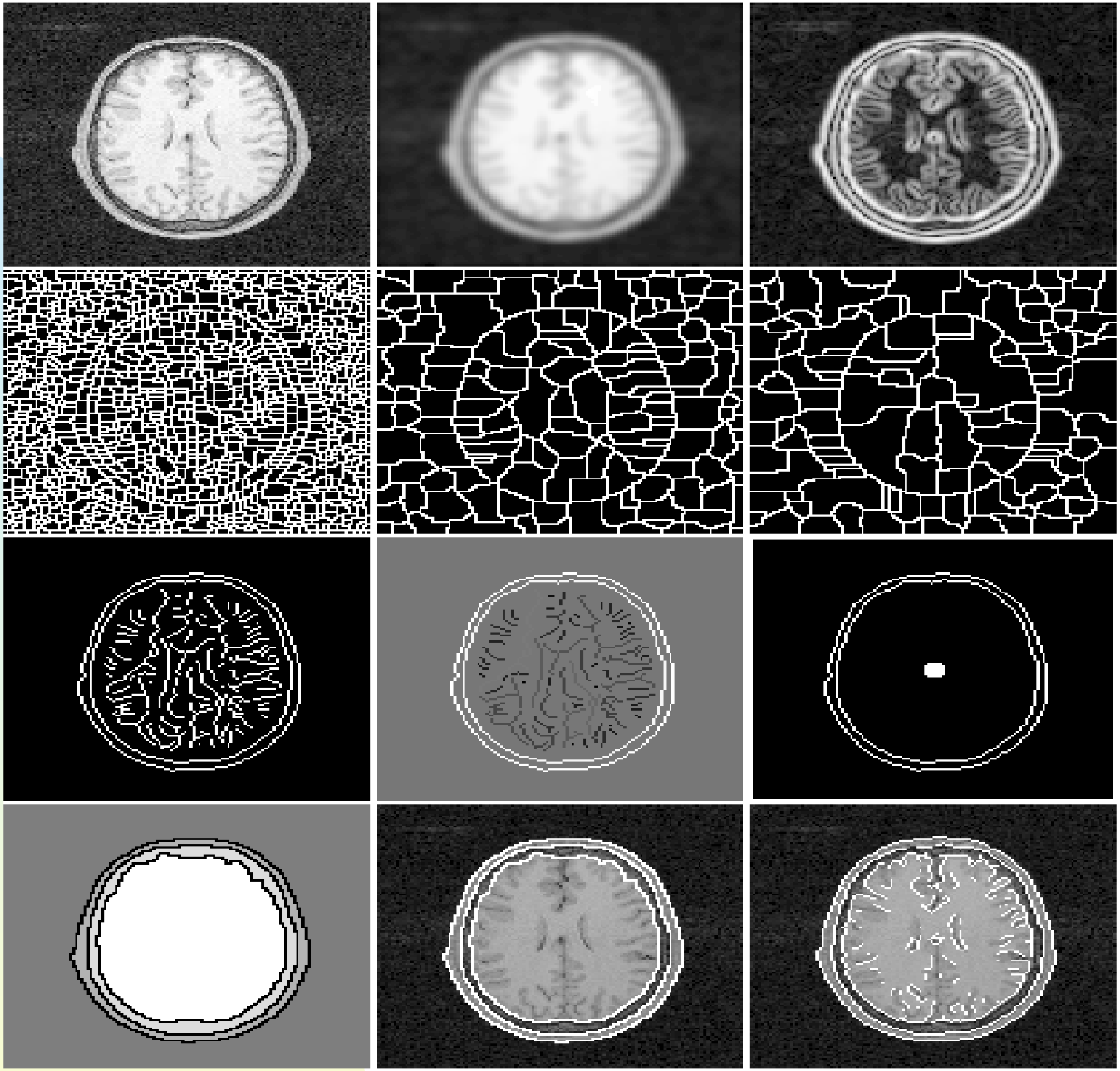


Restored map of Black sea (#coef=9.4%)

Restored map of Black sea (#coef=5.1%)

Restored map of Black sea (#coef=2.8%)

Restored map of Black sea (#coef=1.5%)



Lopez (1998):www.dai.ed.ac.uk/CVonline/LOCAL_COPIES/LOPEZ

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