

A proof of a Dodecahedron conjecture for distance sets

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Abstract

A finite subset of a Euclidean space is called an s -distance set if there exist exactly s values of the Euclidean distances between two distinct points in the set. In this paper, we prove that the maximum cardinality among all 5-distance sets in \mathbb{R}^3 is 20, and every 5-distance set in \mathbb{R}^3 with 20 points is similar to the vertex set of a regular dodecahedron.

Key words: Distance sets, dodecahedron.

1 Introduction

For $X \subset \mathbb{R}^d$, let

$$A(X) = \{d(x, y) \mid x, y \in X, x \neq y\},$$

where $d(x, y)$ is the Euclidean distance between x and y . We call X an s -distance set if $|A(X)| = s$. Two s -distance sets are said to be *isomorphic* if there exists a similar transformation from one to the other. One of the major problems in the theory of distance sets is to determine the maximum cardinality $g_d(s)$ of s -distance sets in \mathbb{R}^d for given s and d , and classify distance sets in \mathbb{R}^d with $g_d(s)$ points up to isomorphism. An s -distance set X in \mathbb{R}^d is said to be *optimal* if $|X| = g_d(s)$. Clearly $g_1(s) = s + 1$, and the optimal s -distance set is the set of $s + 1$ points on the line whose two consecutive points have an equal interval. For the cases where $d = 2$ or $s = 2$, s -distance sets in \mathbb{R}^d are well studied [1, 2, 8, 9, 12, 13, 16, 17, 20], because of their simple structures or the relationship to graphs, see Table 1. For $d \leq 8$, the maximum cardinality $g_d(2)$ are determined, and optimal 2-distance sets in \mathbb{R}^d are classified except for $d = 8$ [8, 13]. Moreover, it is known that $g_3(3) = 12$, $g_3(4) = 13$ and $g_4(4) = 16$, and the classification is complete for the three cases [18, 19]. In particular, we recall the classification of optimal s -distance sets in \mathbb{R}^d for $(d, s) = (2, 4), (3, 3)$ and $(3, 4)$ as in Theorem 1.1.

d	2	3	4	5	6	7	8	s	2	3	4	5	6
$g_d(2)$	5	6	10	16	27	29	45	$g_2(s)$	5	7	9	12	13

Table 1: Maximum cardinalities of s -distance sets in \mathbb{R}^d

Theorem 1.1. ([17, 18, 19])

- (1) Every 9-point 4-distance set in \mathbb{R}^2 is isomorphic to the vertices of the regular nonagon or one of the three configurations given in Figure 1 (a)–(c). Moreover, every 8-point 4-distance set in \mathbb{R}^2 is isomorphic to the vertices of the regular octagon, the vertices of the regular septagon with its center, Figure 1 (d) or 8-point subsets of a 9-point 4-distance set.
- (2) Every 12-point 3-distance set in \mathbb{R}^3 is isomorphic to the vertices of the icosahedron.
- (3) Every 13-point 4-distance set in \mathbb{R}^3 is isomorphic to the vertices of the icosahedron with its center point or the vertex set of the cuboctahedron with its center point.

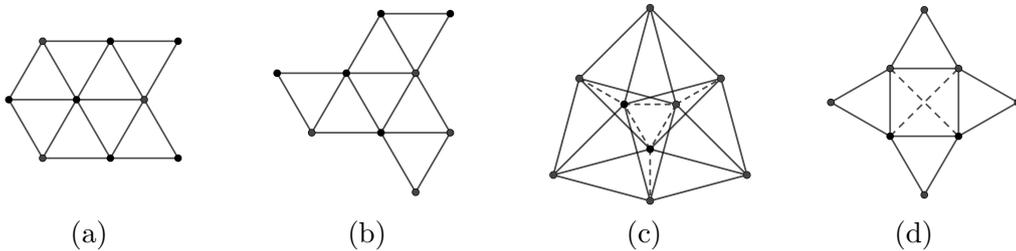


Figure 1: Maximal planar 4-distance sets

For a 2-distance set X , we consider the graph on X where two vertices are adjacent if they have the smallest distance in X . We can construct the 2-distance set that has the structure of a given graph [8]. Lisoněk [13] gave an algorithm for a stepwise augmentation of representable graphs (adding one vertex per iteration), and classified the optimal 2-distance sets in \mathbb{R}^d for $d \leq 7$ by a computer search. Szöllősi and Ostergård [19] extended this algorithm to s -distance sets and classified optimal s -distance sets for $(d, s) = (2, 6), (3, 4), (4, 3)$. Indeed, their algorithm is applicable for small s and d . In the present paper, we add geometrical observations in \mathbb{R}^3 to this algorithm, and obtain the main theorem as follows.

Theorem 1.2. Every 20-point 5-distance set in \mathbb{R}^3 is isomorphic to the vertices of a regular dodecahedron. In particular, $g_3(5) = 20$.

This was a long standing open problem [5] as well as the icosahedron conjecture [8]. The icosahedron conjecture was already solved, and the set is the optimal 3-distance set in \mathbb{R}^3 [18, 19] as in Theorem 1.1 (2). The following theorem plays a key role to prove our main theorem.

Theorem 1.3. Every 5-distance set in \mathbb{R}^3 with at least 20 points contains an s -distance set for some $s \leq 4$ with 8 points.

The main concept to prove Theorem 1.3 is the diameter graph [7] of a subset in \mathbb{R}^d . The diameter graph of a set X in \mathbb{R}^d is the graph on X where two vertices are adjacent if the two vertices have the largest distance in X . The subset of X corresponding to an independence set of the diameter graph does not have the largest distance in X . Thus we can verify the existence of an s' -distance subset of an s -distance set X with $s' < s$ by the independence number of its

diameter graph. The existence of s' -distance set is useful to determine an optimal s -distance set in low dimensions [17, 18]. Ramsey numbers or complementary Ramsey numbers [15] are also expected to show the existence of an s' -distance subsets of an s -distance set.

In section 2, we discuss the distances in a regular dodecahedron and we enumerate the number of 8-point subsets of a dodecahedron which are 3- or 4-distance. In section 3, we consider the independence numbers of diameter graphs and prove Theorem 1.3. The classification of 8-point s -distance sets in \mathbb{R}^3 for $s \leq 4$ are essentially obtained by Szöllősi and Östergård [19]. In section 4, we introduce their methods, where s -distance sets are constructed from s -colorings. In section 5, we classify 8-point 3- or 4-distance sets which may be subsets of a 20-point 5-distance set in \mathbb{R}^3 , and prove Theorem 1.2.

2 Dodecahedron and its subsets

Let $G = (V, E)$ be a simple graph, where $V = V(G)$ and $E = E(G)$ are the vertex set and the edge set of G , respectively. A subset W of $V(G)$ is an *independent set* (*resp. clique*) of G if any two vertices in W are nonadjacent (*resp. adjacent*). The *independence number* $\alpha(G)$ (*resp. clique number* $\omega(G)$) of a graph G is the maximum cardinality among the independent sets (*resp. cliques*) of G . Let $R_i = \{(x, y) \in V \times V \mid \mathfrak{d}(x, y) = i\}$, where \mathfrak{d} is the shortest-path distance. The i -th *distance matrix* A_i of G is the matrix indexed by V whose (x, y) -entry is 1 if $(x, y) \in R_i$, and 0 otherwise. A simple graph G is a *distance-regular graph* [4, 6] if for any non-negative integers i, j, k , the number $p_{ij}^k = |\{z \in V \mid (x, z) \in R_i, (z, y) \in R_j\}|$ is independent of the choice of $(x, y) \in R_k$. The algebra \mathfrak{A} spanned by $\{A_i\}$ over the complex numbers is called the *Bose–Mesner algebra* of a distance-regular graph. There exists another basis $\{E_i\}$ such that $E_i E_j = \delta_{ij} E_i$, where δ_{ij} is the Kronecker delta. The matrices E_i are called *primitive idempotents*, and the matrices are positive semidefinite. The matrices E_i can be interpreted as the Gram matrices of some spherical sets that have the structure of the distance-regular graph, and E_i are called *spherical representations* of the graph. The following matrices

$$P = (p_i(j))_{j,i} \text{ for } A_i = \sum_j p_i(j) E_j,$$

$$Q = (q_i(j))_{j,i} \text{ for } E_i = \frac{1}{|V|} \sum_j q_i(j) A_j,$$

are called the *first* and *second eigenmatrices*, respectively. The entries of P are the eigenvalues of A_i , and the entries of Q are the inner products of the spherical representation of E_i . The first row $q_i(0)$ of Q is the rank of E_i , that is the dimension where the representation E_i exists.

Let \mathcal{D}_{20} be the vertex set of the dodecahedron with edge length 1. The set \mathcal{D}_{20} is a 5-distance set, and let $d_1 = 1, d_2, d_3, d_4, d_5$ be the 5-distances of \mathcal{D}_{20} with $1 = d_1 < d_2 < d_3 < d_4 < d_5$. The second-smallest distance d_2 is the length of a diagonal line of a face, namely $d_2 = \tau = (1 + \sqrt{5})/2$. Since \mathcal{D}_{20} contains the cube with edge length τ , the other distances in the cube are $d_3 = \sqrt{2}\tau$ and $d_5 = \sqrt{3}\tau$. We can calculate $d_4 = \sqrt{3\tau^2 - 1} = \tau + 1$ by Pythagorean theorem. Let \mathfrak{G} be the dodecahedron graph $\mathfrak{G} = (V, E)$, where $V = \mathcal{D}_{20}$ and $E = \{(x, y) \mid d(x, y) = d_1\}$. The graph \mathfrak{G} is a distance-regular graph, and $d(x, y) = d_i$ if and

only if $\mathfrak{d}(x, y) = i$ for each $i \in \{0, 1, \dots, 5\}$, where $d_0 = 0$. The second eigenmatrix Q of \mathfrak{G} is

$$Q = \begin{pmatrix} 1 & 3 & 3 & 4 & 4 & 5 \\ 1 & \sqrt{5} & -\sqrt{5} & -8/3 & 0 & 5/3 \\ 1 & 1 & 1 & 2/3 & -2 & -5/3 \\ 1 & -1 & -1 & 2/3 & 2 & -5/3 \\ 1 & -\sqrt{5} & \sqrt{5} & -8/3 & 0 & 5/3 \\ 1 & -3 & -3 & 4 & -4 & 5 \end{pmatrix}.$$

There are two representations E_2 and E_3 in the 3-dimensional sphere. Indeed, both E_2 and E_3 are the dodecahedron, and the two graphs of A_1 and A_4 are isomorphic. Let Φ be the field automorphism of $\mathbb{Q}(\sqrt{5})$ such that $\Phi(\sqrt{5}) = -\sqrt{5}$ and Φ fixes all rationals. For a matrix $M = (m_{ij})$ with $m_{ij} \in \mathbb{Q}(\sqrt{5})$, the map $\hat{\Phi}(M)$ is defined by applying Φ to the entries of M , namely $\hat{\Phi}(M) = (\Phi(m_{ij}))$. It follows that

$$\begin{aligned} \hat{\Phi}(E_2) &= 3A_0 + \Phi(\sqrt{5})A_1 + A_2 - A_3 - \Phi(\sqrt{5})A_4 - 3A_5 \\ &= 3A_0 - \sqrt{5}A_1 + A_2 - A_3 + \sqrt{5}A_4 - 3A_5 = E_3. \end{aligned}$$

A principal submatrix T of E_2 corresponds to a subset of the dodecahedron. The matrix $\hat{\Phi}(T)$ is a principal submatrix of E_3 , and $\hat{\Phi}(T)$ also corresponds to a subset of the dodecahedron. The two matrices T and $\hat{\Phi}(T)$ may not be isomorphic as distance sets, but the two colorings of them are equivalent (see Section 4 for colorings). This observation gives the following lemma.

Lemma 2.1. *Let X be a subset of the dodecahedron in the unit sphere S^2 . Let M be the Gram matrix of X . Let $\hat{\Phi}$ is the map defined as above. Then M and $\hat{\Phi}(M)$ are subsets of the dodecahedron, and the two colorings of them are equivalent.*

Now we discuss 8-point subsets of the dodecahedron which have only 3 or 4 distances.

Lemma 2.2. *There exists unique 3-distance subset of a regular dodecahedron with 8 points up to isomorphism. The subset is the cube.*

Proof. Let \mathfrak{G} be the dodecahedron graph with relations $R_i = \{(x, y) \in V \times V \mid d(x, y) = d_i\}$. We define the graphs $\mathfrak{G}_i = (V, R_i)$ and $\mathfrak{G}_{i,j} = (V, R_i \cup R_j)$. If for given i , the independence number $\alpha(\mathfrak{G}_{i,j})$ is less than 8 for each $j \neq i$, then we should take the distance d_i for a 8-point subset. We can determine $\alpha(\mathfrak{G}_2) = 6$ and $\alpha(\mathfrak{G}_3) = 5$. This implies $\alpha(\mathfrak{G}_{2,i}) \leq 6 < 8$ and $\alpha(\mathfrak{G}_{3,i}) \leq 5 < 8$ for each $i = 1, 4, 5$. Thus X has both d_2 and d_3 . Moreover, we can determine $\alpha(\mathfrak{G}_{1,5}) = \alpha(\mathfrak{G}_{4,5}) = 7$ and $\alpha(\mathfrak{G}_{1,4}) = 8$, which are calculated by a computer aid. Therefore the distances of X are d_2 , d_3 and d_5 . The set X corresponding to $\alpha(\mathfrak{G}_{1,4})$ is the cube. \square

Lemma 2.3. *There exist exactly 116 of 4-distance subsets of a regular dodecahedron with 8 points up to isomorphism.*

Proof. An 8-point 4-distance subset of a regular dodecahedron does not contain an antipodal pair $\{x, -x\}$, otherwise X is not 4-distance. A regular dodecahedron has only 10 antipodal pairs. We choose 8 antipodal pairs from the 10 pairs, and pick out one point from each antipodal pair, then an 8-point 4-distance set is obtained. Every 8-point 4-distance subset of a regular dodecahedron is obtained by this manner.

First we prove that if two 8-point 4-distance sets X and Y are isomorphic, then X and Y are in the same orbit of the isometry group of a regular dodecahedron. Since the sets

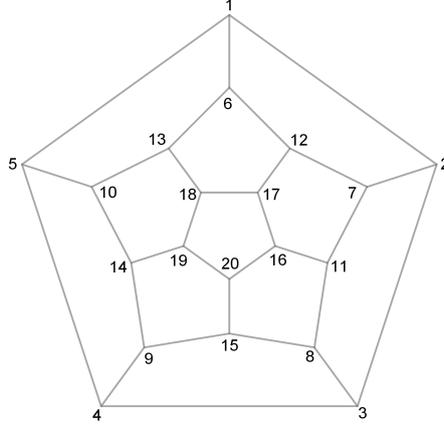


Figure 2: Dodecahedron graph

X and Y are in the same sphere, there exists an isometry σ in the orthogonal group $O(3)$ such that $X^\sigma = Y$. This implies that $(\pm X)^\sigma = \pm Y$, namely the set of 8 antipodal pairs of $\pm X$ are isomorphic to that of $\pm Y$. Since a regular dodecahedron in a given sphere is uniquely determined after one face is fixed, if each set of 8 antipodal pairs makes a face of the dodecahedron, then σ becomes an isometry of the dodecahedron. In order to prove that each set of 8 antipodal pairs makes a face of the dodecahedron, we prove that it is impossible to break all the faces of a regular dodecahedron by removing 2 antipodal pairs. If we remove an antipodal pair, then 6 faces are broken. By removing one more antipodal pair, we would like to break the remaining 6 faces, but it is impossible. Therefore, each set of 8 antipodal pairs contains a face of the dodecahedron, and σ becomes an isometry of the dodecahedron.

We can determine the number of the 4-distance sets up to isomorphism by Burnside's lemma. The isometry group $\text{Aut}(\mathcal{D}_{20})$ of a regular dodecahedron is a subgroup of a symmetric group S_{20} on the 20 vertices, which is isomorphic to $A_5 \times C_2$. The vertices are indexed as Figure 2. Let N_σ denote the number of the 4-distance sets fixed by $\sigma \in \text{Aut}(\Gamma)$. For each $\sigma \in \text{Aut}(\Gamma)$, we determine the number N_σ .

The identity e fixes all the 4-distance sets, namely $N_e = \binom{10}{2} \cdot 2^8 = 11520$.

The transformations that fix a face of the dodecahedron are conjugates of

$$\sigma_1 = (1\ 2\ 3\ 4\ 5)(6\ 7\ 8\ 9\ 10)(11\ 15\ 14\ 13\ 12)(16\ 20\ 19\ 18\ 17),$$

σ_1^2 , σ_1^3 , or σ_1^4 . The number of the transformations is 24. The size of a subset fixed by σ_1 is divisible by 5. Thus $N_{\sigma_1} = 0$, and similarly $N_\sigma = 0$ for any transformation σ in this case.

The transformations that fix an edge of the dodecahedron are conjugates of

$$\sigma_2 = (1\ 2)(3\ 6)(5\ 7)(4\ 12)(10\ 11)(8\ 13)(9\ 17)(14\ 16)(15\ 18)(19\ 20).$$

The number of the transformations is 15. A 4-distance set fixed by σ_2 contains one of $\{1, 2\}$ and $\{19, 20\}$, one of $\{3, 6\}$ and $\{15, 18\}$, one of $\{5, 7\}$ and $\{14, 16\}$, and one of $\{4, 12\}$ and $\{9, 17\}$. This implies that $N_{\sigma_2} = 2^4 = 16$, and similarly $N_\sigma = 16$ for any transformation σ in this case.

The transformations that fix a vertex of the dodecahedron are conjugates of

$$\sigma_3 = (2\ 5\ 6)(3\ 10\ 12)(4\ 13\ 7)(8\ 14\ 17)(9\ 18\ 11)(15\ 19\ 16)$$

or σ_3^2 . The number of the transformations is 20. The size of a subset fixed by σ_3 is congruent to 0 or 1 modulo 3. Thus $N_{\sigma_3} = 0$, and similarly $N_\sigma = 0$ for any transformation σ in this case.

Let τ be the transformation such that $\tau(x) = -x$ for any vertex x , namely

$$\tau = (1\ 20)(2\ 19)(3\ 18)(4\ 17)(5\ 16)(6\ 15)(7\ 14)(8\ 13)(9\ 12)(10\ 11).$$

Clearly $N_\tau = 0$.

We consider the transformations that are conjugates of

$$\tau\sigma_1 = (1\ 19\ 3\ 17\ 5\ 20\ 2\ 18\ 4\ 16)(6\ 14\ 8\ 12\ 10\ 15\ 7\ 13\ 9\ 11),$$

$\tau\sigma_1^2$, $\tau\sigma_1^3$, or $\tau\sigma_1^4$. The number of the transformations is 24. The size of a subset fixed by $\tau\sigma_1$ is divisible by 10. Thus $N_{\tau\sigma_1} = 0$, and similarly $N_\sigma = 0$ for any transformation σ in this case.

We consider the transformations that are conjugates of

$$\tau\sigma_2 = (1\ 19)(2\ 20)(3\ 15)(4\ 9)(5\ 14)(6\ 18)(7\ 16)(12\ 17).$$

The number of the transformations is 15. A 4-distance set fixed by $\tau\sigma_2$ may contain one of $\{1, 19\}$ and $\{2, 20\}$, one of $\{3, 15\}$ and $\{6, 18\}$, one of $\{5, 14\}$ and $\{7, 16\}$, one of $\{4, 9\}$ and $\{12, 17\}$, one of 8 and 13, or one of 10 and 11. This implies that $N_{\tau\sigma_2} = 2^4 + \binom{4}{3}2^5 = 144$, and similarly $N_\sigma = 144$ for any transformation σ in this case.

We consider the transformations that are conjugates of

$$\tau\sigma_3 = (1\ 20)(2\ 16\ 6\ 19\ 5\ 15)(3\ 11\ 12\ 18\ 10\ 9)(4\ 8\ 7\ 17\ 13\ 14)$$

or $\tau\sigma_3^2$. The number of the transformations are 20. A subset fixed by $\tau\sigma_3$ must contain $-x$ for its point x . Thus $N_{\tau\sigma_3} = 0$, and similarly $N_\sigma = 0$ for any transformation σ in this case.

By Burnside's lemma, the number of 8-point 4-distance subsets of the dodecahedron is

$$\frac{1}{|\text{Aut}(\Gamma)|} \sum_{\sigma \in \text{Aut}(\Gamma)} N_\sigma = \frac{1}{120} (1 \cdot 11520 + 24 \cdot 0 + 15 \cdot 16 + 20 \cdot 0 + 1 \cdot 0 + 24 \cdot 0 + 15 \cdot 144 + 20 \cdot 0) = 116. \quad \square$$

3 Diameter graphs and their independence numbers

We denote a path and a cycle with n vertices by P_n and C_n , respectively. We denote a complete graph of order n by K_n . For $X \subset \mathbb{R}^d$, the *diameter* of X is defined to be the maximum value of $A(X)$. Diameters give us important information when we study distance sets especially in few dimensional space. The *diameter graph* $DG(X)$ of $X \subset \mathbb{R}^d$ is the graph with X as its vertices and where two vertices $p, q \in X$ are adjacent if $d(p, q)$ is the diameter of X . Let R_n be the set of the vertices of a regular n -gon. Clearly $DG(R_{2n+1}) = C_{2n+1}$ and $DG(R_{2n}) = n \cdot P_2$. Note that if the independence number $\alpha(DG(X)) = n'$ for an s -distance set X , then the subset of X corresponding to an independence set of order n' is an s' -distance set for some $s' < s$.

For diameter graphs for \mathbb{R}^2 , we have the following propositions [17].

Proposition 3.1. *Let $G = DG(X)$ for $X \subset \mathbb{R}^2$. Then*

- (1) G contains no C_{2k} for any $k \geq 2$;

(2) if G contains C_{2k+1} , then any two vertices in $V(G) \setminus V(C_{2k+1})$ are not adjacent and every vertex not in the cycle is adjacent to at most one vertex of the cycle.

Moreover, G contains at most one cycle.

Proposition 3.2. *Let $G = DG(X)$ be the diameter graph of $X \subset \mathbb{R}^2$ with $|X| = n$. If $G \neq C_n$, then we have $\alpha(G) \geq \lceil \frac{n}{2} \rceil$.*

Propositions 3.1 and 3.2 are implied from the fact that two segments with the diameter must cross if they do not share an end point.

For the diameter graphs of sets in \mathbb{R}^3 , Dol'nikov [7] proved the following theorem. This theorem plays a key role of the proof of Theorem 1.3.

Theorem 3.3 (Dol'nikov). *Let $G = DG(X)$ be the diameter graph of $X \subset \mathbb{R}^3$. If G contains two cycles with odd lengths, then they have a common vertex.*

In particular, we have the following corollary.

Corollary 3.4. *Let $G = DG(X)$ be the diameter graph of $X \subset \mathbb{R}^3$ with $|X| = n$. If G contains an odd cycle C with length m , then $\alpha(G) \geq \lceil \frac{n-m}{2} \rceil$.*

Proof. If we remove the odd cycle C from G , then any odd cycle in G is broken by Theorem 3.3. This implies $G - C$ is a bipartite graph. Therefore we have $\alpha(G) \geq \alpha(G - C) \geq \lceil \frac{n-m}{2} \rceil$. \square

In the remaining of this section, we give a proof of Theorem 1.3.

By Corollary 3.4, if the diameter graph $G = DG(X)$ of $X \subset \mathbb{R}^3$ with 20 points contains a 3-cycle or a 5-cycle, then $\alpha(G) \geq 8$. Let \mathcal{G}_n be the set of all graphs of order n which do not contain neither a 3-cycle nor a 5-cycle. We define

$$f(n) = \min\{\alpha(G) \mid G \in \mathcal{G}_n\}.$$

Since $\alpha(C_n) = \lceil n/2 \rceil$, we have $f(n) \leq \lceil n/2 \rceil$. For a group G and $S \subset G$, we define the *Cayley graph* $\text{Cay}(G, S)$ as the graph whose vertex set is G and two vertices $v, w \in G$ are adjacent if $v^{-1}w \in S$. It is easy to see that $\text{Cay}(\mathbb{Z}_{17}, \{\pm 1, \pm 6\})$ does not contain neither a 3-cycle nor a 5-cycle and $\alpha(\text{Cay}(\mathbb{Z}_{17}, \{\pm 1, \pm 6\})) = 7$. This implies $f(17) \leq 7$.

Lemma 3.5. *Let $f(n)$ be defined as above. Then $0 \leq f(n+1) - f(n) \leq 1$ holds.*

Proof. Let $G \in \mathcal{G}_n$ be a graph satisfying $\alpha(G) = f(n)$ and G' be the graph given by adding one isolated vertex to G . Then $f(n+1) \leq \alpha(G') = \alpha(G) + 1 \leq f(n) + 1$. Let $G \in \mathcal{G}_{n+1}$ be a graph satisfying $\alpha(G) = f(n+1)$ and H be an independent set of G with $|H| = f(n+1)$. Let $v \in V(G) \setminus H$. Then H is an independent set of $G - \{v\}$, and $\alpha(G - \{v\}) = |H|$. Therefore $f(n) \leq \alpha(G - \{v\}) = |H| = f(n+1)$. \square

For a vertex $v \in V(G)$, $\Gamma_i(v) = \{w \in V(G) \mid \mathfrak{d}(v, w) = i\}$, where $\mathfrak{d}(v, w)$ is the shortest-path distance between v and w . We abbreviate $\Gamma(v) = \Gamma_1(v)$. We define $G_i(v)$ as the induced subgraph with respect to $\Gamma_i(v)$ and $k_i(v) = |\Gamma_i(v)|$. Let m be a positive integer. We denote $\Gamma_m^*(v) = \bigcup_{i \geq m} \Gamma_i(v)$ and $k_m^*(v) = |\Gamma_m^*(v)|$. Moreover, we define $G_m^*(v)$ as the induced subgraph of G with respect to $\Gamma_m^*(v)$. Note that we regard $\mathfrak{d}(v, w) = \infty$ and $w \in \Gamma_m^*(v)$ if there is no path between v and w . Then the following degree condition holds.

Lemma 3.6. *Let n and t be positive integers. Let $G \in \mathcal{G}_n$ and $v \in V(G)$. If $\alpha(G) < t < n - k_1(v) + 1$, then*

$$k_1(v) + f(n - k_1(v) - t + 1) < t.$$

Proof. Since $G \in \mathcal{G}_n$, $\{v\} \cup \Gamma_2(v)$ is an independent set of G . In particular, we have $k_3^*(v) \geq n - k_1(v) - t + 1$ since $1 + k_2(v) \leq \alpha(G) < t$ and $k_2(v) \leq t - 2$. Then

$$t > \alpha(G) \geq k_1(v) + f(k_3^*(v)) \geq k_1(v) + f(n - k_1(v) - t + 1),$$

since w_1 and w_2 are not adjacent for any $w_1 \in \Gamma_1(v)$ and $w_2 \in \Gamma_3^*(v)$. \square

For a small integer n , we can determine $f(n)$ by using Lemma 3.6

Lemma 3.7. *We have $f(3) = 2, f(5) = 3, f(8) = 4, f(10) = 5, f(13) = 6$ and $f(17) = 7$.*

Proof. Since $P_3, P_5, C_8, C_{10}, C_{13}$ and $\text{Cay}(\mathbb{Z}_{17}, \{\pm 1, \pm 6\})$ are examples whose independence numbers are the values t in the assertion. This implies the inequalities $f(n) \leq t$ for each case. It is enough to prove the converse inequalities $f(n) \geq t$. We only prove $f(17) \geq 7$ because other inequalities can be proved by a similar way. Suppose that there exists $G \in \mathcal{G}_{17}$ such that $\alpha(G) < 7$. If there exists $v \in V(G)$ such that $k_1(v) = 3$, then $7 > 3 + f(8) = 3 + 4$ from Lemma 3.6 with $t = 7$, which is a contradiction. It is easy to see that we conclude a contradiction for $k_1(v) > 3$ since $k_1(v) + f(n - k_1(v) - t + 1) \leq (k_1(v) + 1) + f(n - (k_1(v) + 1) - t + 1)$ holds in general from $f(n + 1) - f(n) \leq 1$ in Lemma 3.5. Therefore $k_1(v) \leq 2$ for any $v \in V(G)$. Then G is the union of cycles, paths or isolated vertices. Except for an odd cycle, each connected component of G with m vertices has an independent set of size $\lceil m/2 \rceil$. Moreover, G contains at most one odd cycle by Theorem 3.3. Then $\alpha(G) \geq 8$, which is a contradiction. Therefore $f(17) \geq 7$ holds. Then we have $f(17) = 7$. \square

We can determine other $f(n)$ for small n . For example we have $f(12) = 5$ and $f(16) = 7$. For $f(12) = 5$, it is clear because $5 = f(10) \leq f(12)$ and $\text{Cay}(\mathbb{Z}_{12}, \{\pm 1, 6\})$ has the independence number 5. For $f(16) = 7$, it is proved by a similar way to the proof of $f(17) = 7$, and an attaining graph is obtained by removing one vertex from $\text{Cay}(\mathbb{Z}_{17}, \{\pm 1, \pm 6\})$ while maintaining the independence number. However, the values in Lemma 3.7 are enough to prove Lemmas 3.8 and 3.9.

Lemma 3.8. *Let G be the diameter graph of $X \subset \mathbb{R}^3$ with $|X| = 20$. If G is disconnected, then $\alpha(G) \geq 8$.*

Proof. Since G is disconnected, there exists a partition $V = V_1 \cup V_2$ such that v_1 and v_2 are not adjacent for any $v_1 \in V_1$ and $v_2 \in V_2$. We may assume $|V_1| \geq 10$. Let $n_i = |V_i|$ and H_i be the induced subgraph of G with respect to V_i for $i = 1, 2$. Note that we may assume that both H_1 and H_2 do not contain neither a 3-cycle nor a 5-cycle by Corollary 3.4. If $10 \leq n_1 \leq 15$, then $\alpha(H_1) \geq f(n_1) \geq f(10) \geq 5$ and $\alpha(H_2) \geq f(n_2) \geq f(5) \geq 3$ by Lemma 3.7. Then $\alpha(G) = \alpha(H_1) + \alpha(H_2) \geq 5 + 3 = 8$. If $n_1 = 16$, then $\alpha(G) \geq f(16) + f(4) \geq f(15) + f(3) = 6 + 2 = 8$ since $|H_1| \geq 16$ and $|H_2| \geq 4$. If $17 \leq n_1 \leq 19$, then $\alpha(G) \geq f(17) + f(1) \geq 7 + 1 = 8$ since $|H_1| \geq 17$. Therefore $\alpha(G) \geq 8$. \square

Lemma 3.9. *Let G be the diameter graph of $X \subset \mathbb{R}^3$ with $|X| = 20$. Then $\alpha(G) \geq 8$.*

Proof. Suppose that G contains a 3-cycle or a 5-cycle. Then $\alpha(G) \geq 8$ holds by Corollary 3.4. Therefore we may assume that G does not contain neither a 3-cycle nor a 5-cycle. Let $v \in V(G)$. Since G does not contain neither a 3-cycle nor a 5-cycle, both $\Gamma_1(v)$ and $\Gamma_2(v) \cup \{v\}$ are independent sets. Therefore we may assume $k_1(v) \leq 7$ and $k_2(v) \leq 6$. In particular, we may assume $k_3^*(v) = 20 - (1 + k_1(v) + k_2(v)) \geq 13 - k_1(v)$. Moreover, we may assume that G is connected by Lemma 3.8.

Suppose $5 \leq k_1(v) \leq 7$ for some $v \in V(G)$. Since $k_3^*(v) \geq 13 - k_1(v) \geq 6$, we have $\alpha(G_3^*(v)) \geq f(6) \geq f(5) = 3$. Let H be an independent set of $G_3^*(v)$ with $|H| = 3$. Then $\Gamma_1(v) \cup H$ is an independent set of G . Therefore we have $\alpha(G) \geq k_1(v) + f(5) \geq 5 + 3 = 8$. Suppose $k_1(v) = 4$ for some $v \in V(G)$. Since $k_3^*(v) = 13 - k_1(v) \geq 9$, we have $\alpha(G) \geq k_1(v) + f(9) \geq k_1(v) + f(8) = 4 + 4 = 8$. Suppose $k_1(v) = 3$ for some $v \in V(G)$. Then we have $k_3^*(v) = 13 - k_1(v) \geq 10$. Therefore $\alpha(G) \geq k_1(v) + f(10) = 3 + 5 = 8$. Suppose $k_1(w) \leq 2$ for any $w \in V(G)$. Since $k_1(w) \leq 2$ and G is connected, G is isomorphic to C_{20} or P_{20} . Then we have $\alpha(G) = 10$. This completes the proof. \square

By Lemma 3.9, we have Theorem 1.3.

4 Colorings and their realizations

Let $X = \{p_1, p_2, \dots, p_n\}$ be an s -distance set with $A(X) = \{\alpha_1, \alpha_2, \dots, \alpha_s\}$. Let $[n] = \{1, 2, \dots, n\}$ and $\binom{S}{k} = \{T \subset S \mid |T| = k\}$ for a finite set S . An s -distance set with n points is represented by an edge coloring of the complete graph K_n by s colors. We regard an s -coloring of the edge set of K_n by a surjection $c : \binom{[n]}{2} \rightarrow [s]$. We define an s -coloring $c : \binom{[n]}{2} \rightarrow [s]$ of an s -distance set X by a natural manner, namely, $c(\{i, j\}) = k$ where $d(p_i, p_j) = \alpha_k$. Conversely, an s -distance set X is called a *realization* of c if c is a coloring of X .

Two s -colorings c_1 and c_2 are said to be *equivalent* if there exists bijections $g : [s] \rightarrow [s]$ and $h : [n] \rightarrow [n]$ such that $g(c_1(\{h(i), h(j)\})) = c_2(\{i, j\})$ for each $\{i, j\} \in \binom{[n]}{2}$. We define the *coloring matrix* $C = C(x_1, x_2, \dots, x_s)$ of the coloring c with respect to x_1, x_2, \dots, x_s by

$$C_{i,j} = \begin{cases} 0 & \text{if } i = j, \\ x_{c(\{i,j\})} & \text{if } i \neq j. \end{cases}$$

In particular, $C = C(1, 2, \dots, s)$ is called a *normal coloring matrix*. A coloring c is often represented as its normal coloring matrix C in this paper. We distinguish them by lowercase letter c and uppercase letter C .

For a subset $X = \{p_1, p_2, \dots, p_n\} \subset \mathbb{R}^d$, we define the *squared distance matrix* $D = D(X)$ of X by

$$D = (d(p_i, p_j)^2)_{1 \leq i, j \leq n}.$$

For an $n \times n$ symmetric matrix $M = (m_{i,j})_{1 \leq i, j \leq n}$, $\text{Gram}(M)$ is defined to be the $(n-1) \times (n-1)$ symmetric matrix with (i, j) entries

$$\frac{m_{i,n} + m_{j,n} - m_{i,j}}{2}.$$

For $X = \{p_1, p_2, \dots, p_n\}$, the matrix $\text{Gram}(D(X))$ is the Gram matrix of X when p_n is located at the origin.

Theorem 4.1. *Let M be an $(n-1) \times (n-1)$ real symmetric matrix. There exists X in \mathbb{R}^d such that M is equal to the Gram matrix of X if and only if $M = (m_{i,j})_{1 \leq i,j \leq n-1}$ satisfies*

$$\begin{cases} M \text{ is positive semidefinite,} \\ \text{rank } M \leq d, \\ m_{i,i} > 0 \text{ for every } i \in [n-1] \text{ and} \\ m_{i,j} < \frac{m_{i,i} + m_{j,j}}{2} \text{ for every } 1 \leq i < j \leq n-1. \end{cases} \quad (4.1)$$

An s -coloring $c : \binom{[n]}{2} \rightarrow [s]$ is said to be *representable* in \mathbb{R}^d if there exists distinct real numbers $\alpha_1, \alpha_2, \dots, \alpha_s$ such that $C(\alpha_1, \alpha_2, \dots, \alpha_s)$ satisfies (4.1). To decide an s -coloring are not representable, the rank condition in (4.1) is effective. An s -coloring $c : \binom{[n]}{2} \rightarrow [s]$ is said to be *quasi representable* in \mathbb{R}^d if there exists distinct complex numbers $\alpha_1, \alpha_2, \dots, \alpha_s$ such that $\text{rank } C(\alpha_1, \alpha_2, \dots, \alpha_s) \leq d$.

For a square matrix $M = (m_{i,j})_{1 \leq i,j \leq n}$ and an index set $T = \{t_1, t_2, \dots, t_k\} \in \binom{[n]}{k}$, we define a *principal submatrix* of M with respect to T by

$$\text{sub}(M; T) = (m_{t_i, t_j})_{1 \leq i,j \leq k}.$$

Let

$$\mathcal{M}_k(M) = \left\{ \text{sub}(M; T) \mid T \in \binom{[n]}{k} \right\}.$$

We define

$$r(M) = \max \{k \in [n] \mid \exists S \in \mathcal{M}_k, \det S \neq 0\}.$$

Proposition 4.2. *For a square matrix M , $r(M) \leq \text{rank } M$ holds. Moreover, $r(M) = \text{rank } M$ holds if M is positive semidefinite.*

Proof. It is well known that $\text{rank } M$ is the maximum value k such that there exists a square submatrix S of size k in M with $\det S \neq 0$ that may not be principal. This implies $r(M) \leq \text{rank } M$. Suppose M is a positive semidefinite matrix of size n . Since M is positive semidefinite, there exists $n \times \text{rank } M$ matrix N such that $M = NN^\top$ and $\text{rank } N = \text{rank } M$. For $T \in \binom{[n]}{k}$, let χ_T be the $n \times n$ diagonal matrix with diagonal entries $(\chi_T)_{ii} = 1$ if $i \in T$, and $(\chi_T)_{ii} = 0$ if $i \notin T$. For a row vector $\mathbf{x} \in \mathbb{R}^n$ and $T \in \binom{[n]}{k}$, it follows that $\mathbf{x}(\chi_T NN^\top \chi_T^\top) \mathbf{x}^\top = 0$ if and only if $\mathbf{x}(\chi_T N) = 0$. Thus,

$$\begin{aligned} \text{rank}(\chi_T NN^\top \chi_T^\top) &= n - \dim\{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{x}(\chi_T NN^\top \chi_T^\top) \mathbf{x}^\top = 0\} \\ &= n - \dim\{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{x}(\chi_T N) = 0\} = \text{rank}(\chi_T N). \end{aligned}$$

For $T \in \binom{[n]}{k}$, it follows that $\det(\text{sub}(M; T)) \neq 0$ if and only if $\text{rank}(\chi_T NN^\top \chi_T^\top) \geq k$. This implies that $r(M)$ is the maximum value k such that $\text{rank}(\chi_T N) = k$, which is $k = \text{rank } N = \text{rank } M$. \square

An s -coloring $c : \binom{[n]}{2} \rightarrow [s]$ is said to be *weakly quasi representable* in \mathbb{R}^d if there exist distinct complex numbers $\alpha_1, \alpha_2, \dots, \alpha_s$ such that $r(C(\alpha_1, \alpha_2, \dots, \alpha_s)) \leq d$. Clearly if an s -coloring c is representable in \mathbb{R}^3 , then c is a (weakly) quasi representable in \mathbb{R}^3 . The following proposition is essentially proved by Szöllösi and Östergård [19], but we should take all submatrices M that may not be principal in their result. Actually, it is enough to use all principal submatrices M to collect our desired colorings.

n	5	6	7	8	9	10	11	12	13	14
# QRC	512	62095	4499	1093	277	59	12	5	2	0

Table 2: Number of quasi representable at most 4-colorings in \mathbb{R}^3 of n points

Proposition 4.3. *An s -coloring $c : \binom{[n]}{2} \rightarrow [s]$ is a weakly quasi representable in \mathbb{R}^3 if and only if the following system of equations in s values*

$$\begin{cases} \det M = 0 & \text{for all } M \in \mathcal{M}_4(\text{Gram}(C(1, x_1, x_2, \dots, x_{s-1}))), \\ 1 + u \prod_{i=1}^{s-1} x_i(x_i - 1) \prod_{1 \leq j < k \leq s-1} (x_j - x_k) = 0 \end{cases} \quad (4.2)$$

has a complex solution.

5 5-distance sets containing 8-point s -distance sets for $s \leq 4$

By Theorem 1.3, to classify 20-point 5-distance sets in \mathbb{R}^3 , it is enough to consider 5-distance sets which contain 8-point s -distance sets in \mathbb{R}^3 for $s \leq 4$. We will prove the following theorem in this section.

Theorem 5.1. *Let Y be an s -distance set in \mathbb{R}^3 with 8 points for $3 \leq s \leq 4$. If $Y \cup Z$ is a 5-distance set in \mathbb{R}^3 with at least 20 points, then $Y \cup Z$ is isomorphic to a regular dodecahedron.*

Note that there exists no 2-distance set with 8 points in \mathbb{R}^3 . In this section, firstly, we consider (weak) quasi representable s -colorings c in \mathbb{R}^3 instead of s -distance sets in \mathbb{R}^3 . Then we consider realizations of c as needed.

Szöllősi and Östergård [19] classified quasi representable s -colorings in \mathbb{R}^3 for $s \leq 4$, see Table 2.

Lemma 5.2 (Szöllősi and Östergård [19]). *There exist exactly 19 quasi representable 3-colorings in \mathbb{R}^3 with 8 vertices and exactly 1074 quasi representable 4-colorings in \mathbb{R}^3 with 8 vertices.*

We denote the set of all quasi representable s -colorings in \mathbb{R}^3 with n vertices by $\mathcal{CG}(n, s)$. By Lemma 5.2, we have $|\mathcal{CG}(8, 3)| = 19$ and $|\mathcal{CG}(8, 4)| = 1074$.

Let $C \in \mathcal{CG}(8, 3) \cup \mathcal{CG}(8, 4)$. We define a graph $G(C) = (V, E)$ with respect to C as follows. By a computer search, we find all vectors $(a_1, a_2, \dots, a_8) \in [5]^8$ such that

$$M = \begin{pmatrix} 0 & a_1 & \cdots & a_8 \\ a_1 & & & \\ \vdots & & C & \\ a_8 & & & \end{pmatrix} \quad (5.1)$$

is a weakly quasi representable s -colorings in \mathbb{R}^3 for $s \leq 5$ by Proposition 4.3. In order to check whether M is weakly quasi representable, first we calculate a Gröbner basis \mathfrak{B} of system (4.2) for C , see [19] about the manner in details. Then, we calculate a Gröbner basis for the union of \mathfrak{B} and the set of the first equations in (4.2) for all sub($M; T$) with $1 \in T$, which determine whether M is weakly quasi representable. Throughout this paper, computer calculations are done with functions of the software Magma [3] and Maple [14].

We regard the set of all vectors satisfying (5.1) as the vertex set V of the graph $G(C)$. Then two vertices $(a_1, a_2, \dots, a_8), (a'_1, a'_2, \dots, a'_8) \in V$ are adjacent if there exists $i \in [5]$ such that

$$\begin{pmatrix} 0 & i & a_1 & \cdots & a_8 \\ i & 0 & a'_1 & \cdots & a'_8 \\ a_1 & a'_1 & & & \\ \vdots & \vdots & & C & \\ a_8 & a'_8 & & & \end{pmatrix} \quad (5.2)$$

is a weakly quasi representable s -coloring in \mathbb{R}^3 for $s \leq 5$. Some special graphs have loops as Lemma 5.3 below. For positive real numbers $\alpha_1, \alpha_2, \dots, \alpha_n$ and a subset $X = \{p_1, p_2, \dots, p_n\} \subset \mathbb{R}^3$ which is not co-linear, there exist at most two points $q \in \mathbb{R}^3$ such that $d(p_i, q) = \alpha_i$ for any $i \in [n]$. In particular, if there exist two points which satisfy this condition, then X is co-planar. By exhaustive computer search, we have the following lemma.

Lemma 5.3. *Let $C \in \mathcal{CG}(8, 3) \cup \mathcal{CG}(8, 4)$. $G(C)$ has a loop if and only if a realization of C is isomorphic to one of the following nine 4-distance sets.*

- (a) *the subset with 8 points of a regular nonagon,*
- (b) *a regular octagon,*
- (c) *the six subsets with 8 points of the set in Figure 1 (a) in Theorem 1.1,*
- (d) *the set in Figure 1 (d) in Theorem 1.1.*

Moreover, there exist two loops only for (d) and there is only one loop for other cases.

For $C \in \mathcal{CG}(8, 3) \cup \mathcal{CG}(8, 4)$, let $\omega^*(C) = \omega(G(C)) + l(G(C))$, where $\omega(G(C))$ is the clique number of $G(C)$ and $l(G(C))$ is the number of loops in $G(C)$. To prove Theorem 5.1, it is enough to classify $C \in \mathcal{CG}(8, 3) \cup \mathcal{CG}(8, 4)$ such that $\omega^*(C) \geq 12$. By exhaustive computer search, we have the following lemma.

Lemma 5.4. (i) *There exists a unique coloring $C \in \mathcal{CG}(8, 3)$ with $\omega^*(C) \geq 12$, which corresponds to the cube. Moreover, $\omega^*(C) = \omega(G(C)) = 12$ and there exists the unique clique of order 12 for the coloring.*

- (ii) *There exist exactly 63 colorings $C \in \mathcal{CG}(8, 4)$ with $\omega^*(C) \geq 12$. Moreover, $\omega^*(C) = \omega(G(C)) = 12$ and there exists the unique clique of order 12 for each coloring among the 63 colorings.*

We classify 8-point s -distance sets for $s \leq 4$ which are realizations of quasi representable s -colorings in Lemma 5.4. If a realization X of a coloring C is a subset of the dodecahedron, then we have another realization $\hat{\Phi}(X)$ of C by Lemma 2.1. The realizations X and $\hat{\Phi}(X)$ may be isomorphic.

Let $C \in \mathcal{CG}(8, 3)$ be the coloring in Lemma 5.4 (i) and X be a realization of C . Then we have $A(X) = \{1, \sqrt{2}, \sqrt{3}\}$ by solving system (4.2). Then it is easy to see that X is a cube.

Let

$$C_1 = \begin{pmatrix} 0 & 1 & 2 & 2 & 1 & 3 & 3 & 1 \\ 1 & 0 & 1 & 2 & 2 & 2 & 3 & 2 \\ 2 & 1 & 0 & 1 & 2 & 1 & 2 & 3 \\ 2 & 2 & 1 & 0 & 1 & 2 & 1 & 3 \\ 1 & 2 & 2 & 1 & 0 & 3 & 2 & 2 \\ 3 & 2 & 1 & 2 & 3 & 0 & 2 & 4 \\ 3 & 3 & 2 & 1 & 2 & 2 & 0 & 4 \\ 1 & 2 & 3 & 3 & 2 & 4 & 4 & 0 \end{pmatrix}.$$

Then $C_1 \in \mathcal{CG}(8, 4)$ is a coloring in Lemma 5.4 (ii). There exist four solutions of system (4.2) for C_1 and there exist exactly four realizations of C_1 up to isomorphism. Let

$$W = \{6, 12, 17, 18, 13, 16, 19, 1\} \quad (5.3)$$

be a subset of the vertex set of the dodecahedron graph as given in Figure 2. The shortest-path distance matrix $(\mathfrak{d}(x, y))_{x, y \in W}$ of W is C_1 , where C_1 is indexed by W using the order of elements in (5.3). Four realizations of C_1 are given in Figure 3. Let $Y_1 = \{A_1, A_2, \dots, A_7\}$, $Y_2 = \{B_1, B_2, \dots, B_7\}$ and

$$X_i = Y_i \cup \{P_i\}, \quad X'_i = Y_i \cup \{P'_i\} \quad (i = 1, 2),$$

where P'_i is the reflection of P_i in the plane π_i for $i = 1, 2$. Then X_1, X_2, X'_1 and X'_2 are all the realizations of C_1 . Both X_1 and X_2 are subsets of the dodecahedron and have the structure of the coloring C_1 , that shows the situation of Lemma 2.1. There exist exactly two solutions of system (4.2) for the other 62 colorings $C \in \mathcal{CG}(8, 4)$.

If $C \in \mathcal{CG}(8, 4)$ is one of the ten colorings obtained from

$$\begin{aligned} &\{2, 7, 9, 10, 13, 16, 17, 18\}, && \{1, 2, 9, 11, 14, 16, 17, 18\}, && \{1, 8, 12, 15, 16, 17, 18, 19\}, \\ &\{3, 4, 6, 12, 13, 16, 19, 20\}, && \{3, 4, 6, 11, 12, 13, 16, 20\}, && \{6, 8, 10, 12, 14, 16, 19, 20\}, \\ &\{3, 11, 13, 14, 15, 16, 19, 20\}, && \{1, 12, 13, 15, 16, 17, 18, 19\}, && \{2, 9, 11, 13, 14, 15, 16, 20\}, \\ &\{1, 12, 14, 15, 16, 17, 18, 19\}, && && \end{aligned}$$

by the above manner, then the two realizations of C are isomorphic. Except for the above 10 colorings and C_1 , each $C \in \mathcal{CG}(8, 4)$ in Lemma 5.4 (ii) has exactly two realizations up to isomorphism. Then we have the following lemma.

Lemma 5.5. (i) *There exists a unique 3-distance set whose coloring is given in Lemma 5.4 (i).*

(ii) *Among the colorings in Lemma 5.4 (ii), we have the following:*

- (a) *There exists exactly one coloring which has four solutions of system (4.2) and the four realizations corresponding to the solutions are not isomorphic to each other.*
- (b) *There exist exactly 52 colorings which have two solutions of system (4.2) and the two realizations corresponding to the solutions are not isomorphic.*
- (c) *There exist exactly 10 colorings which have two solutions of system (4.2) but the two realizations corresponding to the solutions are isomorphic.*

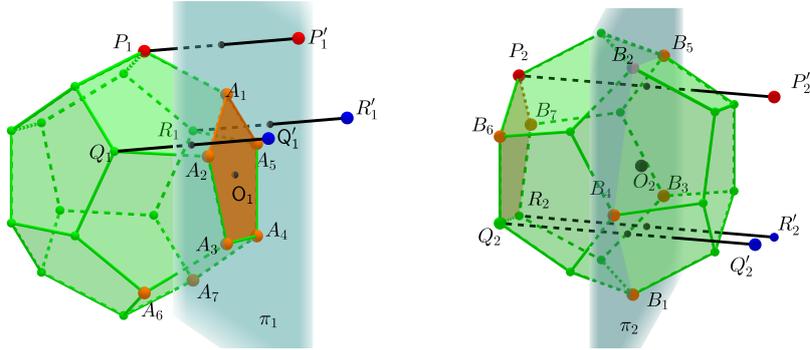


Figure 3: the 8-point subsets which are realizations of C_1

By Lemma 5.5, there exist exactly 118 of 4-distance sets in \mathbb{R}^3 given from the colorings in Lemma 5.4. The two sets X'_1 and X'_2 are not subsets of the dodecahedron. By Lemma 2.3, the remaining 116 4-distance sets should be subsets of the dodecahedron. Note that the cube is also a subset of the dodecahedron. Let S be the set of the cube and the 116 4-distance subsets. By Lemma 5.4, $\omega(G(C)) = 12$ for the coloring C obtained from $X \in S$, and the corresponding clique of order 12 is unique. Therefore a 20-point 5-distance set that contains $X \in S$ must be the dodecahedron.

In order to prove Theorem 5.1, we prove that for $i = 1, 2$ there is no subset $Z \subset \mathbb{R}^3$ such that $X'_i \cup Z$ is a 20-point 5-distance set. We consider a candidate $P \in \mathbb{R}^3$ such that $|A(X'_i \cup \{P\})| \leq 5$ for $i = 1, 2$ by using (5.1). The candidates are $\{Q'_1, R'_1, O_1\}$ for X'_1 and $\{Q'_2, R'_2, O_2\}$ for X'_2 in Figure 3. The points Q'_i and R'_i are the reflections of Q_i and R_i in the plane π_i , respectively. Moreover, O_1 (resp. O'_1) is the center of the pentagon consisted of $\{A_1, A_2, \dots, A_5\}$ (resp. $\{B_1, B_2, \dots, B_5\}$). Thus the cardinality of a 5-distance set that contains X'_i is at most 11. Therefore a proof of Theorem 5.1 is complete.

Finally, we prove Theorem 1.2.

Proof of Theorem 1.2. By Theorem 1.3 and $g_3(2) = 6$, every 5-distance set in \mathbb{R}^3 at least 20 points contains an 8-point s -distance set for some $3 \leq s \leq 4$. Therefore the assertion follows by Theorem 5.1. \square

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