COMPUTER GLUING OF 2D PROJECTIVE IMAGES

G. V. NOSOVSKIY

Faculty Of Mechanics&Mathematics,
Moscow State Univ., Vorob'evy gory,
GSP-2, 119992, Moscow, Russia
E-mail: nosovski@mech.math.msu.su

A method is suggested of computer gluing of 2D projective images of the same object obtained from different points in the space. This problem is well-known in computer geometry [5], [6], [1]. The suggested method is based on a general approach to recognize similar fragments in perturbated sets of objects which was suggested by A.T.Fomenko and the author [3], [4]. The corresponding algorithm has a linear complexity with respect to the total number of pixels in the images and to the number of groups of values which appear in the pixels. *Keywords* : Computer geometry, Multiple view computer geometry, Computer vision, Computer graphics, Projective geometry, Pattern recognition, Stereophotogrammetry, Projective mappings.

1. Introduction

One of the important problems in modern computer geometry is a problem of creating efficient computer algorithms for gluing together 2D projective images of the same object obtained by central projection from different points in the space. Such a problem arises in multiple view geometry, in stereophotogrammetry, etc. See for example [1], [5], [6]. The solution of this problem requires computer algorithms of the recognition of conjugate points. But algorithms of this kind are either not very efficient or require specific additional information which in many cases may not be available.

From purely geometrical point of view this problem could be formulated as a problem of computation of a projective mapping F, which bounds two (in general case - unknown) domains D_1 and D_2, which belong to the same affine coordinate map of a projective plane RP^2. See Figure 1. In order to solve this problem one must have an ability to recognize some pairs of points which correspond to each other by means of an unknown projective mapping F. In applications such points are sometimes called *conjugate points with respect to F* or simply *conjugate points*. We will follow this terminology.

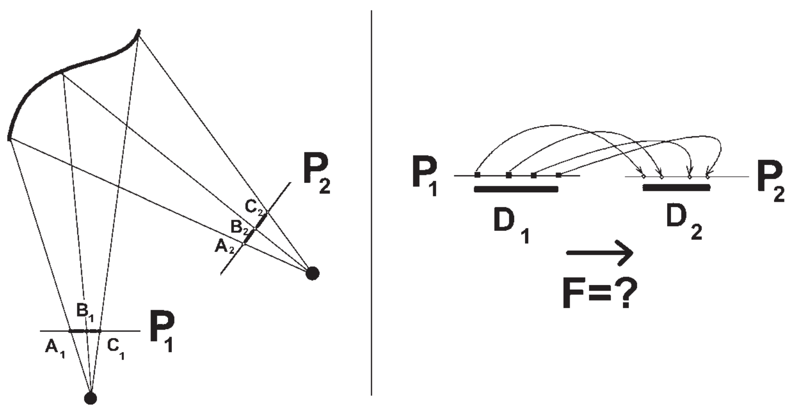

Figure 1.

In this paper we suggest an algorithmic method of computer recognition of conjugate points - mostly for the case when the given 2D images are color (which implies that there is a sufficiently large number of possible values which could appear in the pixels of the images). This method is based on a general approach to the recognition of similar fragments in perturbated sets of objects which was suggested by A.T.Fomenko and the author [3], [4]. The corresponding algorithm has a linear complexity with respect to the total number of pixels in the images and to the number of groups of values which appear in the pixels. We assume that the rule of making such groups was fixed at the very beginning.

It is known from projective geometry that in order to completely define a projective mapping $F : RP^2 \mapsto RP^2$ it is enough to know (absolutely precisely) two sets, each consisting of four arbitrary points in RP^2, such that one of these sets is mapped to another by mapping F. In practical situations, however, one can not determine such points with absolute precision. They are obtained with some error. So, it is necessary to make estimations of the stability of the computed value of F with respect to admissible perturbations of the initial data. Such stability could be characterized by the value of the determinant of the coefficient matrix for the system of linear equations which determine the elements of F. In Section 3 this determinant is calculated which makes it possible to formulate a rule of optimal choice of the configuration of conjugate points.

2. A method of recognition of conjugate points, based on the distribution of shifts between bounded patterns

Assume that we have two photo (or video) images E_1 and E_2, which were obtained by two different cameras. Assume that the same 3D object X appears on both images E_1 and E_2. It means that we have two images of the same object X, which were obtained by central projection from two different points in the space by means of two different, generally non-parallel planes of projection. See Figure 1. The configuration of the centers and planes of projection is unknown. It is also unknown whether the pictures of the object X cover the whole images E_1, E_2 or only some parts of them.

We will call two points (in the discrete case - pixels) of the images E_1 and E_2 respectively *conjugate* if the same point of X was projected to these two points.

Let us assume that both images E_1 and E_2 are digital or were already digitized. Assume that each of them is represented by a square matrix of size $n \times n$. The value of the matrix element (ij) is a vector e_k^{ij} which characterizes the balance of colors and the intensity at the pixel (ij).

Assume that the set of possible values of e_k^{ij} consists of many sufficiently different values. In our case we consider two values as sufficiently different if they can not (except for the cases of influence of random perturbations) appear in conjugate points of images E_1 and E_2.

We assume that the set of possible values of e_k^{ij} was divided into finite quantity m of groups of values close to each other (i.e. we consider grouped sample). We will call these groups *patterns* and denote them by $a_1, \ldots, a_m$. We denote the set of all possible patterns by

$$I = \{a_1, \ldots, a_m\}.$$

Each of the images $E_k (k = 1, 2)$ is represented as an $n \times n$ matrix with pattern numbers $1 \le e_k^{ij} \le m$ as matrix elements.

We will assume that m is large enough. It will be so, for example, in the case of color images. In the case of black and white or grayscale images it is possible to increase the value of m by considering the whole block $d \times d, \quad (d > 1)$ of pixels as an elementary cell of matrix $E_k (k = 1, 2)$.

Our purpose is to build an efficient algorithmic procedure of the recognition of conjugate cells of the images E_1 and E_2 in the situation when there is no apriori information about the object X and its location on the images E_1 and E_2. All information concerning the conjugate cells and their vicinities should be taken from the original images E_1 and E_2 which are ordinary pictures and do not carry any additional facilities for recognition.

In this article we consider the case when the vertical axes of the rectangular images E_1 and E_2 respectively approximately correspond to each other. It will be so, for example, if the columns and rows of both matrices E_1 and E_2 approximately represent horizontal and vertical planes in the space. This is the case for many applications when the cameras are positioned either horizontally or vertically. If not, the algorithm could be modified to take in account a possible rotation.

We will follow the non-parametric approach to the recognition of similar fragments suggested in the works of A.T.Fomenko and the author [3 4].

Assume that the size n of the matrices E_1, E_2 has the form $n = Np$ for some natural numbers N and p. Consider the division of the matrices E_1, E_2 into square blocks of size $p \times p$ containing p^2 matrix elements each. The number of such blocks in E_1 or E_2 is equal to N^2. We will denote them by $E_k^{\alpha\beta}, \quad 1 \leq k \leq 2, \quad 1 \leq \alpha, \beta \leq N:$

$$E_k^{\alpha\beta} = \{e_k^{ij} : (\alpha - 1)p < i \leq \alpha p, (\beta - 1)p < i \leq \beta p\}.$$

In our algorithm we will interpret the blocks $E_k^{\alpha\beta}$ as fragments consisting of pixels sufficiently close to each other. The number p (block size) is a parameter which should be chosen according to the concrete application. We will assume that $p \geq 2$.

Definition 2.1. .
The shift of two cells $e_1^{ij} \in E_1$ and $e_2^{i'j'} \in E_2$ is the vector ρ with two natural numbers as components:

$$\rho(e_1^{ij}, e_2^{i'j'}) = (i' - i, j' - j). \tag{1}$$

Definition 2.2. .
We call two patterns $n_i, n_j \in I$ bounded (with respect to the pair of matrices E_1 and E_2 with fixed division to blocks) if they appeared in the same block of any of these two matrices:

$$\exists k \in \{1,2\}, \exists \alpha, \beta \in \{1, \ldots, N\} : \quad n_i, n_j \in E_k^{\alpha\beta}.$$

If the patterns $n_i, n_j \in I \quad (1 \leq i, j \leq m)$ are bounded we will denote this by $n_i \sim n_j$.

If two cells $e_1^{ij} \in E_1$ and $e_2^{i'j'} \in E_2$ contain a pair of bounded patterns respectively we will also call these cells bounded and denote this by $e_1^{ij} \sim e_2^{i'j'}$.

In particular, any pattern which appeared in any of the images E_1, E_2, is bounded with itself with respect to matrices E_1, E_2,.

We consider the cells and patterns which appear in these cells as elements of the matrices E_1, E_2 (or their blocks) and use the same symbol "$\in$" in both cases.

We will start with the following construction. Using given matrices E_1, E_2 consider a random choice of two cells in E_1 and E_2 respectively. The corresponding probability space $\Omega, \Sigma, \mathbf{P}$ is defined as follows. Let $\Omega = E_1 \times E_2, \Sigma = 2^\Omega$ and $\mathbf{P}$ be the uniform distribution on Ω given by $\mathbf{P}\{\omega\} = 1/n^2$ for any $\omega \in \Omega$. Denote the cell randomly chosen from E_1 by $e_1^{ij}(\omega)$, and the cell randomly chosen from E_2 by $e_2^{i'j'}(\omega)$. Indices $i = i(\omega), j = j(\omega), i' = i'(\omega), j' = j'(\omega)$ are random here but for simplicity we omit the argument ω.

Denote by $b_1 = b_1(\omega)$ the pattern which appeared in the first chosen cell $e_1^{ij}(\omega) \in E_1$, and by $b_2(\omega)$ the pattern which appeared in the second chosen cell $e_2^{i'j'}(\omega) \in E_2$.

Let us define the random variable $\xi = \xi(\omega)$ on $\Omega, \Sigma, \mathbf{P}$ which is the shift between the chosen cells:

$$\xi(\omega) = \rho(e_1^{ij}(\omega), e_2^{i'j'}(\omega)). \tag{2}$$

Then with probability one ξ takes values in the domain of the natural 2-dimensional lattice: $\{(i_1, i_2): \quad 1-n \le i, j \le n-1\}$.

Assume that there exists at least one pair of bounded cells belonging to E_1 and E_2 respectively. This assumption is natural in our case, i.e. when E_1 and E_2 represent two different images of the same object X. In this case, even in the presence of some perturbations, there should exist such a pair of patterns close to each other (i.e. belonging to the same block) in one image that at least one of them appears in another image also. This is enough for existence of bounded pairs of patterns.

This assumption implies that the probability $\mathbf{P}(A)$ of the event A is greater then zero:

$$A = \{\omega : b_1(\omega) \sim b_2(\omega)\}. \tag{3}$$

Event A means that the chosen pair of patterns is bounded (see Definition 2.2).

The first stage of our algorithm of the recognition of conjugate points includes a procedure of elimination of comparison of most pairs of fragments belonging to E_1 and E_2 respectively. This elimination is based on a preliminary analysis of the following two distributions:

$$f_0(i,j) = \mathbf{P}\{\xi(\omega) = (i,j)\}, \tag{4}$$
$$f_1(i,j) = \mathbf{P}\{\xi(\omega) = (i,j)|A\} \qquad (1-n \le i,j \le n-1) \tag{5}$$

Here f_0 denotes the unconditional distribution of the above defined random variable ξ on the natural lattice in $\mathbf{R^2}$, and f_1 denotes its conditional distribution under the condition A defined in (3).

We will need the explicit form of the function f_0. It is given in the following two lemmas.

Lemma 2.1. *For any* $1-n \le i,j \le n-1$ *the following relations hold:*

$$f_0(i,j) = f_0(-i,j) = f_0(i,-j) = f_0(-i,-j). \tag{6}$$

Proof. Denote by V the clockwise rotation of a square matrix to the angle of 90^o :

$$V(e_k^{ij}) = e_k^{j,n+1-i} \quad (1 \le k \le 2, \quad 1 \le i,j \le n).$$

The transformation V is an automorphism on the set of cells of a matrix E_k. Therefore V could be considered as an automorphism of $\Omega = E_1 \times E_2$ which in the discrete case always preserves uniform distribution: $\mathbf{P}\{V\omega\} = \mathbf{P}\{\omega\}$. Furthermore, if both matrices E_1 and E_2 were rotated by the same angle, then the shift vector between any pair of cells belonging to E_1 and E_2 respectively rotates by the same angle. Therefore:

$$f_0(i,j) = \mathbf{P}\left\{\xi(\omega) = (i,j)\right\} = \mathbf{P}\left\{\xi(V\omega) = (i,j)\right\} = \mathbf{P}\left\{\xi(\omega) = (-i,j)\right\}$$
$$= f_0(-i,j).$$

Similarly:

$$f_0(i,j) = \mathbf{P}\left\{\xi(V^2\omega) = (i,j)\right\} = \mathbf{P}\left\{\xi(\omega) = (-i,-j)\right\} = f_0(-i,-j),$$

$$f_0(i,j) = \mathbf{P}\left\{\xi(V^3\omega) = (i,j)\right\} = \mathbf{P}\left\{\xi(\omega) = (i,-j)\right\} = f_0(i,-j).$$

This completes the proof of the lemma.

According to this lemma, in order to find the distribution f_0 it is enough to calculate $f_0(i,j)$ for $i,j \geq 0$. This is done in the following lemma.

Lemma 2.2. *Let* $0 < i,j \leq n-1$. *Then*

$$f_0(i,j) = \mathbf{P}\left\{\xi(\omega) = (i,j)\right\} = \frac{1}{n^2} - \frac{i+j}{n^3} + \frac{ij}{n^4}, \tag{7}$$

$$f_0(i,0) = \frac{1}{n^2} - \frac{i}{n^3}, \tag{8}$$

$$f_0(0,j) = \frac{1}{n^2} - \frac{j}{n^3}, \tag{9}$$

$$f_0(0,0) = \frac{1}{n^2}. \tag{10}$$

The distribution f_0 does not depend on the entries of matrices E_1 and E_2. It is determined by their size n only.

Proof. Let us prove (7). The probability that the shift (i,j) appears between two chosen cells $e_1^{ij}(\omega)$ and $e_2^{i'j'}(\omega)$ according to the uniform distribution $\mathbf{P}$ is equal to the ratio of the number of appropriate chances to the total quantity of chances. The total quantity of chances is equal to n^4 (the number of pairs of cells belonging to E_1 and E_2 respectively). In our case the number of appropriate chances is equal to the number of such cells in E_1, that there exists another cell in E_2 such that its shift from the first one is equal to (i,j). It is clear that if such a cell in E_2 exists then it is defined uniquely by the first one. Notice that the number of cells in a square matrix $n \times n$ such that there exists another cell shifted from it by (i,j), is equal to $n^2 - ni - nj + ij$. Consequently we have

$$f_0(i,j) = \frac{n^2 - ni - nj + ij}{n^4} = \frac{1}{n^2} - \frac{i+j}{n^3} + \frac{ij}{n^4}.$$

Relations (8)-(10) could be proved by similar considerations. This completes the proof of the lemma

Comment. It is easy to see that the function f_0 does not change if the shift vector between the cells $e_1^{ij} \in E_1^{\alpha\beta}$ and $e_2^{i'j'} \in E_1^{\alpha'\beta'}$ is defined by the following formula instead of (1):

$$\widetilde{\rho}(e_1^{ij}, e_2^{i'j'}) = (\alpha' - \alpha, \beta' - \beta'). \tag{11}$$

The difference between f_1 and f_0 has a spike at the values of typical shifts between the conjugate cells of the matrices E_1 and E_2 [3], [4]. This spike could be recognized by standard statistical procedures. In this way the typical shift between conjugate cells is determined.

The suggested algorithm of determining the conjugate cells in the matrices E_1 and E_2 includes two stages. At the first stage the functions f_0 and f_1 are calculated and compared by statistical procedures. As a result of this comparison the typical shifts between bounded patterns (with respect to E_1 and E_2) are determined. They are interpreted as shifts between conjugate cells [3], [4]. Algorithmically, this stage of calculations could be done during one path through the set of the cells of the matrices E_1 and E_2. For fixed N and m the complexity of the corresponding calculations is linear with respect to the total number of cells $2n^2$.

At the second stage the conjugate pairs of pixels are determined finally. At this stage we do not need to compare all pairs of fragments of the matrices E_1 and E_2 respectively. It is enough to consider only pairs of blocks $(E_1^{\alpha\beta}, E_2^{\alpha'\beta'})$, which contain sufficiently many such bounded pairs of cells that they are shifted to the typical values, determined at the first stage. This approach significantly decreases the amount of pairs of fragments in E_1 and E_2 which should be analyzed for the existence of conjugate cells. For the final determination of conjugate cells the standard statistical procedures (which are commonly used in this situation) could be used. But in many applications it is enough just to find the intensity peaks in both fragments under consideration.

It should be stressed that we do not assume that the representations of the object X in E_1, E_2 could be transformed to one another by some shift or, more generally, by some affine mapping. This transformation is assumed to be projective. But according to our assumptions there will still exist a range of typical shifts between conjugate points. See Figure 2. It is because of the assumption that the rows and columns of both matrices E_1, E_2 approximately represent horizontal and approximately vertical planes in the space (consequently images E_1 and E_2 are not rotated by a large angle one with respect to other). Otherwise our algorithm requires a certain modification to take into account the angle of rotation.

Let us discuss briefly the problem of the statistical determination of significant spikes of the difference $(f_1 - f_0)$ – i.e. such spikes which corre-

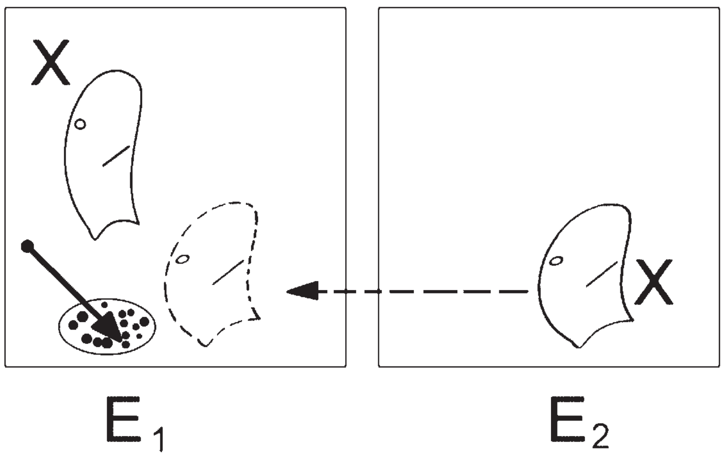

Figure 2.

spond to the typical shifts between conjugate points. To do this we need to eliminate random spikes of the difference $(f_1 - f_0)$.

In order to analyze random spikes let us assume that there are no conjugate points in the given images E_1 and E_2. Then it is natural to assume that E_1 and E_2 are independent random elements defined on some probability space $(\Omega_1, \Sigma_1, \mathbf{P}_1)$. In this case the random variable ξ and event A (see (2), (3)) could be defined on the product $(\Omega_1, \Sigma_1, \mathbf{P}_1) \times (\Omega, \Sigma, \mathbf{P})$. If E_1 and E_2 are independent then ξ is independent from A. Consequently the distributions f_0 and f_1 considered on $\Omega_1 \times \Omega_1$ will coincide. It follows from lemma 2.2 that f_0 in this case will be determined by the same formulas (7)-(10). Consequently, the distribution f_1, defined on the product $\Omega_1 \times \Omega$, will be determined by formulas (7)-(10). For a concrete ω_1 (i.e. conditionally for concrete matrices E_1 and E_2) the distribution f_1 can differ from f_0, but if n is large enough then this difference will be small for a wide class of distributions $\mathbf{P}_1$ (according to the central limit theorem). In applications the value of n is usually greater then 1000 which is enough to make random spikes of the difference $f_1 - f_0$ much smaller than the spike corresponding to conjugate points.

Lemma 2.3 below suggests a possible method to eliminate random spikes of $f_1 - f_0$. We will prove it under the following assumptions.

Assume that the given pair of square matrices (E_1, E_2) of size $n \times n$ is a result of some (arbitrary) stochastic experiment $(\Omega_1, \Sigma_1, \mathbf{P}_1)$. Without loss of generality we will assume that $(E_1, E_2) = \omega_1^0 \in \Omega_1$ is an elementary event of Ω_1 and that an arbitrary $\omega_1 \in \Omega$ could be represented by a pair of square matrices of size $n \times n$ which contain natural numbers in the range $1...m$ as their matrix elements.

Consider the above described construction $(\Omega, \Sigma, \mathbf{P})$ of the uniform random choice of two cells from the matrices $\omega_1 = (E_1, E_2)$ Then on the product $(\Omega_1, \Sigma_1, \mathbf{P}_1) \times (\Omega, \Sigma, \mathbf{P})$ we can define random variable ξ and event A by the same relations (2) and (3).

Let us define the functions f_0, f_1 (i.e. unconditional and conditional distributions of ξ) by the relations similar to (4), (5) with the only difference that we take the product measure $\mathbf{P_1} \times \mathbf{P}$ instead of the measure $\mathbf{P}$. It follows from lemma 2.2 that the function f_0 in this situation will still be defined by the same formulas (7) -(10).

Below ω_1 and ω will denote not only elements of the probability spaces Ω_1 and Ω_1, but also subsets of the form $\omega_1 \times \Omega$ and $\omega \times \Omega_1$ in $\Omega_1 \times \Omega_1$.

Let us fix some $\varepsilon > 0, \quad 1 - n \leq i, j \leq n - 1$ and define an event on $\Omega_1 \times \Omega_1$ by

$$A_{ij}^{\varepsilon} = \{(\omega_1, \omega) : \mathbf{P}\{\xi = (i,j)|A, \omega_1\} - \mathbf{P}\{\xi = (i,j)|A\} \geq \varepsilon\}. \tag{12}$$

Let us define the functions $f_1(i,j) - f_0(i,j)$ by the same formulas (4), (5) as above.

If ξ and A are independent then the event A_{ij}^{ε} consists of such elementary events $(E_1, E_2) \times \omega$, that the difference $f_1(i,j) - f_0(i,j)$, calculated from (E_1, E_2) at the point (i,j) is not less then ε.

Then the following lemma holds.

Lemma 2.3.

Assume that ξ and A are independent. Then $\forall \varepsilon > 0, \quad 1 - n \leq i, j \leq n - 1:$

$$\mathbf{P}\left\{A_{ij}^{\varepsilon}|A\right\} \leq \frac{f_0(i,j)}{f_0(i,j) + \varepsilon}. \tag{13}$$

Proof. Fix some $\varepsilon > 0, \quad 1 - n \leq i, j \leq n - 1$. Denote by B the event $B = \{\xi = (i,j)\}$. According to the assumptions of the lemma, events B and A are independent. Therefore using (12) we have:

$$
\begin{aligned}
\mathbf{P}\{B|A\} &= \frac{\mathbf{P}\{BA\}}{\mathbf{P}\{A\}} = \frac{\sum_{\omega_1}\mathbf{P}\{BA\omega_1\}}{\mathbf{P}\{A\}} = \\
&= \frac{\sum_{\omega_1}\mathbf{P}\{B|A\omega_1\}\mathbf{P}\{A\omega_1\}}{\mathbf{P}\{A\}} = \sum_{\omega_1}\mathbf{P}\{B|A\omega_1\}\mathbf{P}\{\omega_1|A\} \\
&\geq \sum_{\omega_1\subset A_{ij}^{\varepsilon}}\mathbf{P}\{B|A\omega_1\}\mathbf{P}\{\omega_1|A\} \\
&\geq \sum_{\omega_1\subset A_{ij}^{\varepsilon}}(\mathbf{P}\{B|A\}+\varepsilon)\mathbf{P}\{\omega_1|A\} \\
&= (\mathbf{P}\{B|A\}+\varepsilon)\mathbf{P}\left\{A_{ij}^{\varepsilon}|A\right\}.
\end{aligned}
$$

The events B and A are independent, so the above formula implies:

$$
\mathbf{P}\left\{A_{ij}^{\varepsilon}|A\right\} \leq \frac{\mathbf{P}\{B|A\}}{\mathbf{P}\{B|A\}+\varepsilon} = \frac{\mathbf{P}\{B\}}{\mathbf{P}\{B\}+\varepsilon} = \frac{f_0(i,j)}{f_0(i,j)+\varepsilon}. \tag{14}
$$

This completes the proof of the lemma.

3. The stability of determination of the projective mapping using a configuration of conjugate points

Let us recall some notions from projective geometry.

For any nonzero vector $x \in R^3$ denote by $\widehat{x}$ the straight line parallel to x and passing through 0. Any such line determines uniquely a point on the projective plane RP^2 and could be considered as an element of RP^2 : $\widehat{x} \in RP^2$. For an $\widehat{x} \in RP^2$ its *homogeneous coordinates* are coordinates of x in R^3. The multiplication of homogeneous coordinates by an arbitrary coefficient $\lambda \neq 0$ does not change the point in RP^2. Homogeneous coordinates are denoted by $(x_1 : x_2 : x_3)$. The affine coordinates of a point $\widehat{x} \in RP^2$ are defined as follows. Fix some plane π in R^3 such that $0 \notin \pi$ and fix some basis on it. We say that $\widehat{x} \in RP^2$ belongs to the affine map π if and only if the line $\widehat{x}$ intersects the plane π in R^3. The affine coordinates of the point $\widehat{x} \in RP^2$ in the affine map π are defined as the coordinates of their point of intersection in the basis which was fixed on π.

Assume that some orthonormal basis $\{e_1, e_2, e_3\}$ was fixed in R^3. We will fix the affine map S_3 which is generated by the plane $x_3 = 1$ with basis $\{e_1, e_2\}$ on it. Without loss of generality we assume that the domains which are mapped into one another by the unknown projective mapping F belong to the map S_3. The affine coordinates in S_3 of a point $\widehat{x} = (x_1 : x_2 : x_3) \in S_3$ are $(x_1/x_3, x_2/x_3)$.

Consider four points in the space RP^2. We will say that they are *in general position* if and only if any three of them represented by straight lines in R^3 passing through 0, do not belong to any plane.

Theorem 3.1. *(see. for example [2]). Assume that $\{P_1, P_2, P_3, P_4\}$ and $\{Q_1, Q_2, Q_3, Q_4\}$ are two sets of four points from RP^2 each. Assume that both these sets are in general position in RP^2. Then there exists a unique projective mapping such that it maps P_i into Q_i for all $i = 1, 2, 3, 4$.*

In order to analyze the stability of the computation of this mapping we need an explicit form of a system of linear equations for it.

We will denote a point from RP^2 and any vector corresponding to it in R^3 by a same letter. Assume that two sets of four points $\{P, Q, R, T\} \subset RP^2$ and $\{P', Q', R', T'\} \subset RP^2$ in general position are given and all these points belong to the affine map S_3. Denote by $F = (f_{ij}) \quad (1 \le i, j \le 3)$ the 3×3 square matrix which defines a linear mapping $R^3 \to R^3$ which corresponds to the projective mapping which maps P_i into Q_i for all $i = 1, 2, 3, 4$. Such a matrix is not uniquely defined – it could be multiplied by an arbitrary nonzero coefficient. In order to define it uniquely assume that

$$\begin{gathered} P = (p_1, p_2, 1), \quad Q = (q_1, q_2, 1), \quad R = (r_1, r_2, 1), \quad T = (t_1, t_2, 1), \\ P' = (p'_1, p'_2, 1), \quad Q' = (q'_1, q'_2, q'_3), \quad R' = (r'_1, r'_2, r'_3), \quad T' = (t'_1, t'_2, t'_3) \end{gathered} \tag{15}$$

and

$$F(P) = P', \quad F(Q) = Q', \quad F(R) = R', \quad F(T) = T'. \tag{16}$$

We write:

$$\begin{aligned} a_p &= p'_1, \quad b_p = p'_2, \qquad a_q = \frac{q'_1}{q'_3}, \quad b_q = \frac{q'_2}{q'_3}, \\ a_r &= \frac{r'_1}{r'_3}, \quad b_r = \frac{r'_2}{r'_3}, \qquad a_t = \frac{t'_1}{t'_3}, \quad b_t = \frac{t'_2}{t'_3}. \end{aligned} \tag{17}$$

The four vector relations (16) represent a system of 12 linear scalar equations with 9 unknown coefficients f_{ij} of the matrix F and 3 unknown coordinates q'_3, r'_3, t'_3. It could be written in the following form:

$$A\mathbf{x} = \mathbf{y}, \tag{18}$$

where

$$\mathbf{x} = (f_{11}, f_{12}, f_{13}, f_{21}, f_{22}, f_{23}, f_{31}, f_{32}, f_{33}, q_3', r_3', t_3')^T,$$

$$\mathbf{y} = (a_p, 0, 0, 0, b_p, 0, 0, 0, 1, 0, 0, 0)^T,$$

$$A = \begin{pmatrix} p_1 & p_2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ q_1 & q_2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -a_q & 0 & 0 \\ r_1 & r_2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -a_r & 0 \\ t_1 & t_2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -a_t \\ 0 & 0 & 0 & p_1 & p_2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & q_1 & q_2 & 1 & 0 & 0 & 0 & -b_q & 0 & 0 \\ 0 & 0 & 0 & r_1 & r_2 & 1 & 0 & 0 & 0 & 0 & -b_r & 0 \\ 0 & 0 & 0 & t_1 & t_2 & 1 & 0 & 0 & 0 & 0 & 0 & -b_t \\ 0 & 0 & 0 & 0 & 0 & 1 & p_1 & p_2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & q_1 & q_2 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & r_1 & r_2 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & t_1 & t_2 & 1 & 0 & 0 & -1 \end{pmatrix}$$

The proof of the following lemma is due to E.S.Skripka.

Lemma 3.1. . *The determinant of the matrix A is proportional to the product of the areas of triangles $\triangle PQR, \triangle PRT, \triangle PQT$, and $\triangle Q'R'T'$. The relation holds:*

$$|detA| = 16 S_{\triangle PQR} S_{\triangle PRT} S_{\triangle PQT} S_{\triangle Q'R'T'} \tag{19}$$

Proof. Direct calculation shows that:

$$detA =$$

$$(q_2p_1 - q_1p_2 - r_1q_2 - r_2p_1 + r_2q_1 + r_1p_2)(-t_2p_1 + r_2p_1 + t_2r_1 - t_1r_2 + t_1p_2 - r_1p_2)$$

$\times(q_2p_1-q_1p_2-t_1q_2-t_2p_1+t_2q_1+t_1p_2)(b_ta_q-b_ta_r+b_qa_r+b_ra_t-b_ra_q-b_qa_t)$

Note that the absolute values of the factors on the right hand side of this relation are equal to the areas of following triangles multiplied by 2:

$$|q_2p_1 - q_1p_2 - r_1q_2 - r_2p_1 + r_2q_1 + r_1p_2| = 2S_{\triangle PQR},$$

$$|-t_2p_1 + r_2p_1 + t_2r_1 - t_1r_2 + t_1p_2 - r_1p_2| = 2S_{\triangle PRT},$$

$$|q_2p_1 - q_1p_2 - t_1q_2 - t_2p_1 + t_2q_1 + t_1p_2| = 2S_{\triangle PQT},$$

$$|b_ta_q - b_ta_r + b_qa_r + b_ra_t - b_ra_q - b_qa_t| = 2S_{\triangle Q'R'T'}.$$

Let us prove for example the last formula (see. Figure 3):

$$S_{\triangle R'Q'T'} = |S_{R'Q'DF} + S_{T'R'FE} - S_{T'Q'DE}| =$$

$$= \left|(Q'D + R'F)\,\frac{b_q - b_r}{2} + (T'E + R'F)\,\frac{b_r - b_t}{2} - (T'E + Q'D)\,\frac{b_q - b_t}{2}\right| =$$

$$= \left|\frac{Q'D}{2}\,(b_t - b_r) + \frac{R'F}{2}\,(b_q - b_t) + \frac{T'E}{2}\,(b_r - b_q))\right| =$$

$$= \left|\frac{a_q}{2}\,(b_t - b_r) + \frac{a_r}{2}\,(b_q - b_t) + \frac{a_t}{2}\,(b_r - b_q)\right| =$$

$$= \frac{1}{2}|b_ta_q - b_ta_r + b_qa_r + b_ra_t - b_ra_q - b_qa_t|.$$

The remaining tree formulas can be proved similarly. This completes the proof of the lemma.

it follows from this lemma that in order to perform a robust calculation of the projective mapping with help of given conjugate points one should try to choose such a configuration of conjugate points that the product of the areas of 4 triangles built on them is as large as possible.

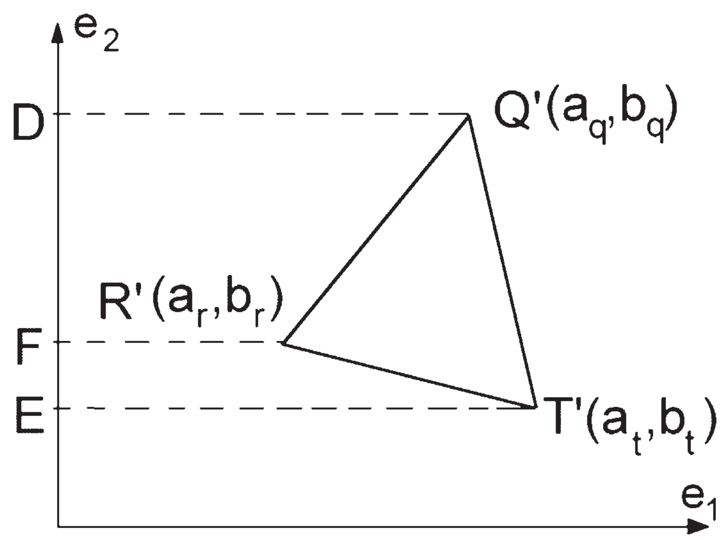

Figure 3.

References

1. "Multiple view computer geometry," Cambridge Univ. Press, Cambridge, 2000.
2. R.Hartshorne, "Foundations of Projective Geometry," Lecture notes Harward Univ., W.A.Benjamin. inc., NY, 1967.
3. A.T.Fomenko,G.V.Nosovskiy, "Recognition of original structures in mixed sequences," *Trudy seminara po vektornomu i tenzornomu analizu*, Vol. 22, Moscow Univ. Publishing House, Moscow, 1985, pp. 119-131 (In Russian).
4. A.T.Fomenko,G.V.Nosovskiy, "Recognition of original structures in mixed sequences," *Trudy seminara po vektornomu i tenzornomu analizu*, Vol. 23, Moscow Univ. Publishing House, Moscow, 1988, pp. 104-121 (In Russian).
5. "Photogrammetry,", Ed. L.N.Kel' et al. "Nedra", Moscow, 1989 (In Russian).
6. B.P.Horn, "Vision of robots," "Mir", Moscow, 1989 (In Russian).