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Non-degenerate Singularities of a Three-dimensional Billiard Bounded by an Ellipsoid in a Hooke Potential Field

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Abstract—Consider the problem of motion of a material point under the action of an elastic force of coefficient k inside a triaxial ellipsoid whose center coincides with the center of the force field. Such a dynamical system is Liouville integrable in the piecewise-smooth sense. For the attractive and repulsive cases ($k > 0$ and $k < 0$, respectively), we describe the Liouville foliation of the system in small neighborhoods of regular layers as well as layers containing nondegenerate critical points.

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1. INTRODUCTION

Currently, the qualitative theory of integrable Hamiltonian systems (hereinafter referred to as IHS) is actively studied. Such systems arise quite often in various problems of physics, mechanics and geometry.

The most geometrically illustrative IHS are integrable billiards and their generalizations. T he integrability of the billiard in the region bounded by an ellipse was noted in the work of G.D. Birkhoff [1]. Billiards in flat regions bounded by arcs of confocal quadrics are also integrable. Such systems started to be studied with respect to the Liouville equivalence in the works of V. Dragovich and M. Radnovich [2, 3] and V.V. Vedyushkina (Fokicheva) [4, 5]. In addition, V.V. Vedyushkina classified all locally flat topological billiards bounded by arcs of confocal ellipses and hyperbolas, as well as regions obtained by gluing elementary regions along convex (see [6]) and arbitrary (see [7, 8]) boundary segments. The billiard table equivalence was defined (see also [9]), and the Fomenko invariants (i.e., rough molecules) and the Fomenko–Zieschang invariants (labeled molecules) were calculated for each nonequivalent billiard table.

Also V.V. Vedyushkina introduced and considered a new class of integrable billiards—billiards on table-books, i.e., CW-complexes glued from elementary flat confocal regions along some common (isometric) sections of boundaries. To each edge of the gluing e there is assigned a permutation σ_e, which defines the dynamics of a material point falling on this edge. The material point moving along the ith sheet of the book and hitting the edge e with a velocity vector v continues its motion from the point of impact with the velocity vector reflected relative to e along the sheet numbered $\sigma_e(i)$. The billiards on table-books realize Liouville foliations of many important IHS. Moreover, according to the results of V.V. Vedyushkina and I.S. Kharcheva, the billiard books model all three-dimensional bifurcations of the IHS (see [10]), and also realize the Liouville foliation bases of all IHS of two degrees of freedom on surfaces of constant energy (see [11]).

In addition to topological billiards and billiard books, there are other integrable generalizations of the classical billiard inside an ellipse. These include billiards with integrable potentials [12–15], billiards in

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the Minkowski plane [16], billiards with slipping [17], evolutionary force billiards [18, 19] and confocal billiards on quadrics [20].

The present paper is devoted to the study of the Liouville foliation topology of a three-dimensional billiard in a Hooke potential field inside an ellipsoid. Note that in the case of two degrees of freedom, a similar problem was studied by I.F. Kobtsev in [12, 13] and S.E. Pustovoitov in [14, 15]. The classical three-dimensional billiard inside an ellipsoid (i.e., the system in absence of external forces) was considered by V. Dragovich and M. Radnovich in [3]. In this paper, they constructed a bifurcation diagram of such a billiard and described the regular layers of the Liouville foliation. G.V. Belozerov in [21] described regular layers and their 1-bifurctions for all three-dimensional billiards bounded by confocal quadrics. Homeomorphism classes of the isoenergetic surfaces of such systems were found.

Previously, the author has found homeomorphism classes of nonsingular surfaces of constant energy of the billiard inside a triaxial ellipsoid in a Hooke potential field (see [22]). The purpose of this paper is to study the Liouville foliation structure of this system near regular layers as well as layers containing nondegenerate critical points. In the next paragraph we give the explicit form of the first integrals of this billiard and prove formulae of separating variables. Then in the third paragraph we describe the regions of possible motion and construct the bifurcation diagrams. In the fourth paragraph we describe Liouville foliation in the neighborhood of regular layers. After that in the fifth paragraph we study the topology of the Liouville foliation in the neighborhood of layers containing nondegenerate critical points. Note that the method of work described in the final paragraph is based on the property of conservation of a part of the action variables at bifurcations of non-zero rank. A.T. Fomenko used a similar method to describe the fundamental properties of 3-atoms (see [23]), and N.T. Zung to prove the theorem on the semi-local form of nondegenerate singularities of the IHS (see [24, 25]).

2. STATEMENT OF THE PROBLEM. INTEGRABILITY

Let $\mathcal{E}$ be an ellipsoid in $\mathbb{R}^3$ given by the equation $\dfrac{x^2}{a}+\dfrac{y^2}{b}+\dfrac{z^2}{c}=1$, where $a>b>c$. Consider the following dynamical system. A material point of unit mass moves inside the region D bounded by the ellipsoid $\mathcal{E}$ in a Hooke potential field of coefficient k. We assume that the center of the force field coincides with the origin. We will assume that the reflection from the boundary of $\mathcal{E}$ is absolutely elastic. Such a dynamical system will be called by *billiard in a Hooke potential field inside the ellipsoid*, and the region D by *billiard table*.

Let us describe the phase space of this system. To do this, consider the subset $\widehat{M}^6=\{(x,v)\in T\mathbb{R}^3|\ x\in D, v\neq 0\}$ of $T\mathbb{R}^3$ with induced topology. Due to the billiard reflection we need to identify all pairs (x_1,v_1), (x_2,v_2) of $\widehat{M}^6$ such that

$$x_1=x_2\in\mathcal{E},\quad ||v_1||=||v_2||,\quad v_1-v_2\perp T_{x_1}\mathcal{E}.$$

Let us denote this equivalence relation on $\widehat{M}^6$ by $\sim$. The set $M^6=\widehat{M}^6/\sim$ with the factor-topology is the phase space of the system under consideration.

Note that one of the first integrals of our billiard is the total mechanical energy

$$H=\frac{1}{2}\big(\dot{x}^2+\dot{y}^2+\dot{z}^2\big)+\frac{k}{2}\big(x^2+y^2+z^2\big).$$

This function is continuous on M^6.

The phase space M^6 due to the reflection from the boundary is, in general, not a smooth manifold. We denote by $\widetilde{M}^6$ the union of smooth pieces of M^6. Note that on $\widetilde{M}^6$ the form $\omega=dv_x\wedge dx+dv_y\wedge dy+dv_z\wedge dz$ (here v_x,v_y,v_z are the components of a velocity vector in Cartesian coordinates) is correctly defined. It has the continuous limit at breakpoints of M^6 (i.e., on the boundary of the table D). For such systems, A.T. Fomenko introduced the notion of *integrability by Liouville in the piecewise-smooth sense*. We will say that on M^6 there is a Liouville integrable Hamiltonian system in the piecewise-smooth sense with a Hamiltonian function H if

1. the dynamics a the point on $\widetilde{M}^6$ is defined by the vector field
$$H = \omega^{ij} \frac{\partial H}{\partial x^i} \frac{\partial}{\partial x^j};$$

2. there exist continuous on M^6 and smooth on $\widetilde{M}^6$ functions F_1 and F_2 such that the set H, F_1, F_2 is involutive and functionally independent on $\widetilde{M}^6$.

Remark 1. In some cases, it is possible to introduce a smooth structure at all points of the "break" of the manifold M^6 so that the form ω and the functions H, F_1, and F_2 also become smooth. The question of smoothing the manifold M^6 was considered by V.F. Lazutkin [30] and E.A. Kudryavtseva [31].

Proposition 1. *The billiard in a Hooke potential field inside an ellipsoid is an integrable Hamiltonian system in the piecewise-smooth sense. The additional first integrals of this system are the following*
$$F_1 = \frac{c+b}{2}\dot{x}^2 + \frac{a+c}{2}\dot{y}^2 + \frac{a+b}{2}\dot{z}^2$$
$$-\frac{1}{2}B\left(K_x^2 + K_y^2 + K_z^2\right) + \frac{k}{2}\left((b+c)x^2 + (a+c)y^2 + (a+b)z^2\right),$$
$$F_2 = \frac{cb}{2}\dot{x}^2 + \frac{ac}{2}\dot{y}^2 + \frac{ab}{2}\dot{z}^2 - \frac{1}{2}\left(aK_x^2 + bK_y^2 + cK_z^2\right) + \frac{k}{2}\left(bcx^2 + acy^2 + abz^2\right).$$
Here K_x, K_y, and K_z are components of the angular momentum vector.

Proof. Actually this fact follows from the classical result of Jacobi about the integrability of the geodesic flow on a fouraxial ellipsoid in an elastic force field (see [26]). Nevertheless, we will find the explicit form of the additional first integrals of our billiard. To do this we will use the method described by V.V. Kozlov in [27]. Namely we will search for additional first integrals F_1 and F_2 of our system in the form $F_1 = I_1 + f_1$, $F_2 = I_2 + f_2$, where I_1 and I_2 are additional quadratic first integrals of the problem without the potential, and f_1 and f_2 are functions depending on spatial variables only. It is convenient to take the following additional first integrals of the billiard without the potential
$$I_1 = \frac{c+b}{2}\dot{x}^2 + \frac{a+c}{2}\dot{y}^2 + \frac{a+b}{2}\dot{z}^2 - \frac{1}{2}\left(K_x^2 + K_y^2 + K_z^2\right),$$
$$I_2 = \frac{cb}{2}\dot{x}^2 + \frac{ac}{2}\dot{y}^2 + \frac{ab}{2}\dot{z}^2 - \frac{1}{2}\left(aK_x^2 + bK_y^2 + cK_z^2\right).$$
Here K_x, K_y, and K_z are components of the angular momentum vector.

Note that if F_1 and F_2 are the first integrals of the problem without reflection, then these functions will be the first integrals of our billiard. Indeed, under the reflection I_1 and I_2 do not change since they are the first integrals of the problem without potential, and the functions f_1 and f_2 will not change too since they depend on the spatial variables only.

Equating to zero the time derivatives of F_1 and F_2 and using Newton's equations of the problem without reflection, we obtain
$$-k((c+b)x\dot{x} + (a+c)y\dot{y} + (a+b)z\dot{z}) + \partial_x f_1 \dot{x} + \partial_y f_1 \dot{y} + \partial_z f_1 \dot{z} = 0,$$
$$-k(cbx\dot{x} + acy\dot{y} + abz\dot{z}) + \partial_x f_2 \dot{x} + \partial_y f_2 \dot{y} + \partial_z f_2 \dot{z} = 0.$$
The variables in the equations are separated and we can easily find their partial solutions
$$f_1 = \frac{k}{2}\left((b+c)x^2 + (a+c)y^2 + (a+b)z^2\right),$$
$$f_2 = \frac{k}{2}\left(bcx^2 + acy^2 + abz^2\right).$$
So, the additional first integrals of our system have the required form.

It remains to prove that the functions H, F_1, and F_2 are functionally independent and the equality $\{F_1, F_2\} = 0$ is true. The latter fact is verified by direct calculations. The functional independence of

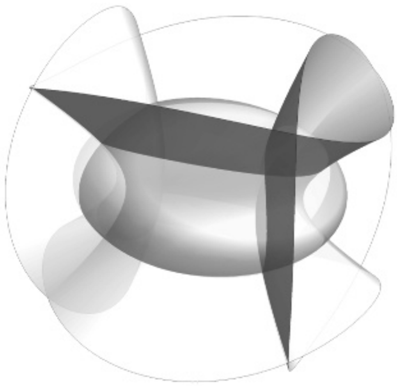

Fig. 1. Three non-degenerate confocal quadrics.

H, F_1, and F_2 follows from the following considerations. It turns out that the determinant $\dfrac{D(T, I_1, I_2)}{D(\dot{x}, \dot{y}, \dot{z})}$, where T is the kinetic energy of the material point, does not equal zero almost everywhere. Since the potential energy and the functions f_1, f_2 depend on spatial variables only, the equality $\dfrac{D(H, F_1, F_2)}{D(\dot{x}, \dot{y}, \dot{z})} = \dfrac{D(T, I_1, I_2)}{D(\dot{x}, \dot{y}, \dot{z})}$ holds. □

Studying the integrability of the geodesic flow on an ellipsoid Jacobi showed that in elliptic coordinates the variables in the geodesic equations are separable. The same property holds for the geodesic equations on an ellipsoid in an elastic force field [26]. Therefore, it is very reasonable to rewrite the equations of motion of our system in elliptic coordinates. Let us recall their definition. To do this, let us associate the family of confocal quadrics with the ellipsoid $\mathcal{E}$.

Definition 1. *A family of confocal quadrics in three-dimensional Euclidean space* is the set of quadrics given by the equation

$$(b-\lambda)(c-\lambda)x^2 + (a-\lambda)(c-\lambda)y^2 + (a-\lambda)(b-\lambda)z^2 = (a-\lambda)(b-\lambda)(c-\lambda), \qquad (1)$$

where $a > b > c > 0$ are fixed numbers, and λ is a real parameter. If the parameter of a quadric of the family is equal to a, b or c, then the quadric is called degenerate, otherwise it is called non-degenerate.

Remark 2. Note that degenerate quadrics are coordinate planes. If $\lambda \in (-\infty, c)$, then the corresponding quadric is an ellipsoid, if $\lambda \in (c, b)$, then it's a one-sheeted hyperboloid, if $\lambda \in (b, a)$, then it's a two-sheeted hyperboloid (see Fig. 1). The quadric of parameter 0 is the ellipsoid $\mathcal{E}$.

The family of confocal quadrics has many remarkable properties. Let us list some of them.

1. The tangent planes at the points of intersection of two confocal quadrics are orthogonal.

2. Through each point $P \in \mathbb{R}^3$ pass exactly three confocal quadrics (considering multiplicity). If P does not lie in any of the coordinate planes, then one of these quadrics is an ellipsoid, the second is a one-sheeted hyperboloid, and the third is a two-sheeted hyperboloid.

If we map to each point in $\mathbb{R}^3$ a triple of numbers $(\lambda_1, \lambda_2, \lambda_3)$, where λ_1, λ_2, and λ_3 are the ascending-ordered parameters of the confocal quadrics passing through this point, then we obtain a coordinate system that in each coordinate octant will be single-valued, smooth, and regular.

Definition 2. *The triple of the functions $(\lambda_1, \lambda_2, \lambda_3)$ is called the elliptic coordinates in $\mathbb{R}^3$.*

Note that $\lambda_1 \in (-\infty, c]$, $\lambda_2 \in [c, b]$, and $\lambda_3 \in [b, a]$.

Proposition . *In the elliptic coordinates, the variables of our system are separated and the equations of motion in them are the following*

$$\dot{\lambda}_i = \pm\frac{2\sqrt{2}}{(\lambda_i-\lambda_j)(\lambda_i-\lambda_k)}\sqrt{V(\lambda_i)}. \tag{2}$$

Here $i = 1, 2, 3$, $V(z) = \left(f_2 - zf_1 + z^2h - \frac{k}{2}(a-z)(b-z)(c-z)\right)(a-z)(b-z)(c-z)$, *and* f_1, f_2, *and* h *are values of the first integrals* F_1, F_2, *and* H, *respectively.*

We omit the proof of this proposition since it is technical. However, the formula of separating variables 2 is very important for describing the topology of the Liouville foliation of the system.

3. REGIONS OF POSSIBLE MOTION. BIFURCATION DIAGRAMS

To describe the regions of possible motion, we need to study the properties of the polynomial $V(z)$ from the Proposition 2. Indeed, if for given values f_1, f_2, h and a point with elliptic coordinates $(\lambda_1, \lambda_2, \lambda_3)$ the inequalities $V(\lambda_i) \geqslant 0$, $i = 1, 2, 3$ are true, then at least one velocity vector is defined at this point. And if at least one of the inequalities is not true, then no trajectory passes through this point at the common level of the first integrals.

The following proposition describes the structure of the roots of the polynomial $V(z)$.

Proposition 3. *The polynomial* $V(z)$ *has* 6 *real roots taking multiplicity into account.*

Proof. We know that the elliptic coordinates satisfy the restrictions $\lambda_1 \in (-\infty; c]$, $\lambda_2 \in [c, b]$, and $\lambda_3 \in [b, a]$. The points a, b, c are the roots of $V(z)$. Therefore, in the case of general position inside the intervals $(-\infty; c]$, $[c, b]$, and $[b, a]$ there must be found sub-intervals A_1, A_2, and A_3, respectively, inside which the polynomial $V(z)$ takes positive values and vanishes on the boundary. The total number of boundary points of the intervals A_i is either five or six.

Note that the intervals A_i and A_j can intersect at either a, b, or c only. But in this case, the polynomial $V(z)$ to the right and left of the intersection point of A_i and A_j is positive. So, this point is a multiple root of $V(z)$. It follows that $V(z)$ has at least 5 real roots, each of which is a boundary point of the interval A_i. Since the degree of the polynomial $V(z)$ is six, all its roots are real. □

Let ξ_1, ξ_2, and ξ_3 be roots of the polynomial $f_2 - zf_1 + z^2h - \frac{k}{2}(a-z)(b-z)(c-z)$, where (h, f_1, f_2) is a nonempty common level of the first integrals (H, F_1, F_2). As we have just proved, all these numbers are real. Let us assume that $\xi_1 \leqslant \xi_2 \leqslant \xi_3$.

Note that ξ_1, ξ_2, and ξ_3 are roots of the polynomial depending on the values of the first integrals H, F_1, and F_2 and constants a, b, c, and k. So, ξ_1, ξ_2, and ξ_3 are the first integrals of our problem. Note that the first integrals ξ_1, ξ_2, and ξ_3 are connected to F_1, F_2, and H by Vyet's formulae

$$\begin{cases} (a+b+c) - \dfrac{2H}{k} = \xi_1 + \xi_2 + \xi_3, \\ (ab+bc+ca) - \dfrac{2F_1}{k} = \xi_1\xi_2 + \xi_2\xi_3 + \xi_1\xi_3, \\ abc - \dfrac{2F_2}{k} = \xi_1\xi_2\xi_3. \end{cases}$$

The condition $\xi_1 \leqslant \xi_2 \leqslant \xi_3$ guarantees a one-to-one correspondence between the allowed values of the triples (H, F_1, F_2) and (ξ_1, ξ_2, ξ_3). It follows from this fact and Vyet's formulae that ξ_1, ξ_2, and ξ_3 are functionally independent commuting first integrals of our system.

Using the new first integrals, let us describe typical regions of possible motion (i.e., of general position) of the billiard under consideration. According to the proof of the Proposition 3, the roots of $V(z)$ uniquely describe the structure of the regions of possible motion. If we fix ξ_1, ξ_2, and ξ_3, then on those intervals where the polynomial $V(z)$ was positive at $k < 0$, it will become negative at $k > 0$ and vice versa. Therefore, the cases of attractive $(k > 0)$ and repulsive $(k < 0)$ potentials are very different from each other. Let us consider each of them separately.

Attractive potential. In the case of general position when $k > 0$ the following 8 types of regions of possible motion (hereinafter referred to as RPM) of a material point in $\mathbb{R}^3$ are feasible.

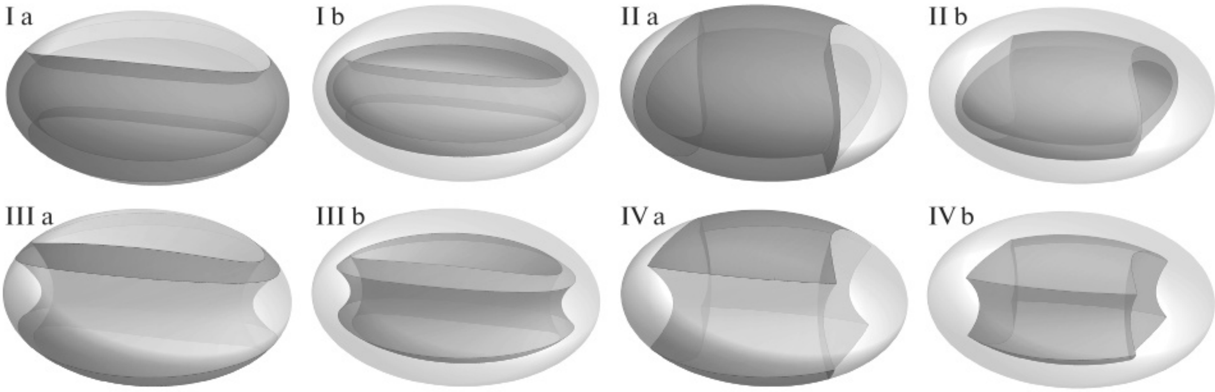

Fig. 2. Regions of possible motion of the billiard inside an ellipsoid in a Hooke attractive potential field.

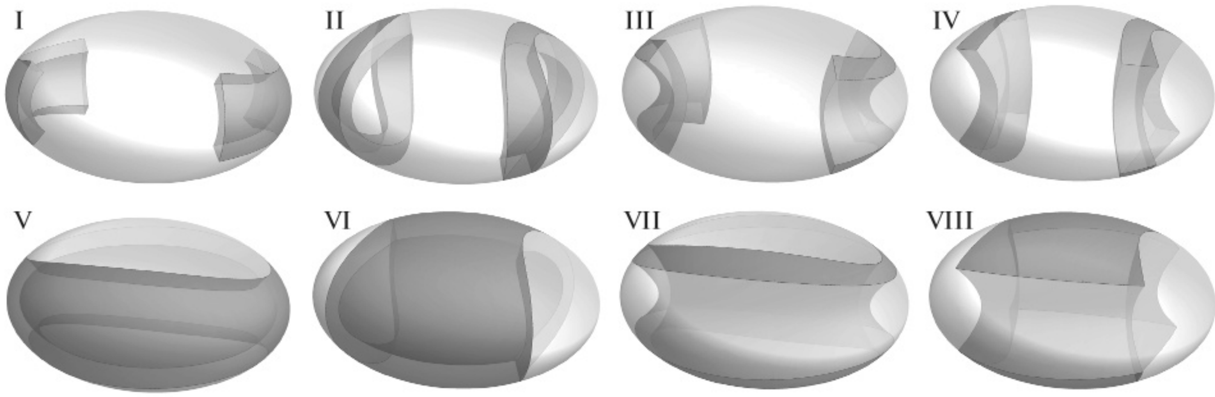

Fig. 3. Regions of possible motion of the billiard inside an ellipsoid in a Hooke repulsive potential field.

All of these regions are shown in Fig. 2.

Note that regions I a and I b, II a and II b, III a and III b, IV a and IV b differ from each other only by the following fact. The elliptical boundary of the RPM passes from the boundary of the billiard table to the quadric lying strictly inside it. This means that in regions of type i a the particle reflects from the outer elliptical wall of the RPM, and in i b touches it. However, reflection and touching from the point of view of "behavior" of the system are equivalent. Hence the regions of form i a and i b can be combined into one class. Thus, at $k > 0$ there are exactly 4 different types of regions of possible motion, which we will number with Roman numerals from I to IV without attributing Latin letters.

Repulsive potential. In the case of general position when $k > 0$, the following 8 types of regions of possible motion of a material point in $\mathbb{R}^3$ are feasible.

All of these regions are shown in Fig. 3.

Now, using the classification of regions of possible motion, let us construct bifurcation diagrams of our billiards for $k > 0$ and $k < 0$. Despite the fact that the integrals H, F_1, and F_2 are quadratic and are computed quite easily, it more convenient to work with the set of integrals ξ_1, ξ_2, and ξ_3.

Consider the momentum mapping $\mathcal{F} : M^6 \mapsto \mathbb{R}^3(\xi_1, \xi_2, \xi_3)$. Since the billiard system is piecewise-smooth, and the integrals ξ_i cease to be smooth at points where one of the conditions of the form $\xi_i = \xi_j$ when $i \neq j$ is true (nevertheless, they remain continuous), we independently introduce the notions of the critical value of the momentum mapping and the bifurcation diagram.

The types of RPMs define the stratification of the image of the mapping $\mathcal{F}$. Note that the type of RPM changes if and only if the polynomial $V(z)$ has a multiple root. In this regard, we conclude the following definition.

Definition 3. *A point $P = (\xi_1, \xi_2, \xi_3)$ of the momentum mapping image will be called critical if one of the following conditions is satisfied:*

Table 1. Classification of typical regions of possible motion of the billiard when $k > 0$.

Region number	Inequalities defining the RPM
I a	$\xi_1 < 0 < \xi_2 < c < \xi_3 < b$
I b	$0 < \xi_1 < \xi_2 < c < \xi_3 < b$
II a	$\xi_1 < 0 < \xi_2 < c < b < \xi_3 < a$
II b	$0 < \xi_1 < \xi_2 < c < b < \xi_3 < a$
III a	$\xi_1 < 0 < c < \xi_2 < \xi_3 < b$
III b	$0 < \xi_1 < c < \xi_2 < \xi_3 < b$
IV a	$\xi_1 < 0 < c < \xi_2 < b < \xi_3 < a$
IV b	$0 < \xi_1 < c < \xi_2 < b < \xi_3 < a$

Table 2. Classification of typical regions of possible motion of the billiard when $k < 0$.

Region number	Inequalities defining the RPM
I	$\xi_1 < c < \xi_2 < b < \xi_3 < a$
II	$\xi_1 < c < b < \xi_2 < \xi_3 < a$
III	$c < \xi_1 < \xi_2 < b < \xi_3 < a$
IV	$c < \xi_1 < b < \xi_2 < \xi_3 < a$
V	$\xi_1 < c < \xi_2 < b < a < \xi_3$
VI	$c < \xi_1 < b < \xi_2 < a < \xi_3$
VII	$c < \xi_1 < \xi_2 < b < a < \xi_3$
VIII	$c < \xi_1 < b < \xi_2 < a < \xi_3$

- The point P belongs to the boundary of the image of the momentum mapping $\mathcal{F}$.
- The polynomial $V(z)$ has a multiple root.

All other points will be called regular.

Definition 4. *The set of all critical points of the momentum mapping image will be called bifurcation diagram.*

Figure 4 shows the bifurcation diagrams of a billiard with attractive and repulsive potentials inside the ellipsoid $\mathcal{E}$. Note that the first diagram has 4 chambers, while the second has exactly 8 chambers. Each of the chambers corresponds to a different type of region of possible motion (see Tables 1 and 2).

Next, we will study the local structure of the Liouville foliation. First of all, we will be interested in layers corresponding to critical points. Therefore, we have to distinguish a class of critical points which are well studyable.

Definition 5. *A critical point $P = (\xi_1, \xi_2, \xi_3)$ is called wrong if there are at least three equal numbers among the numbers ξ_1, ξ_2, ξ_3, a, b, and c. Otherwise, the point P is called right.*

Remark 3. In the following, we will not study wrong points.

Note that this definition of the right point is not chosen by chance. In wrong critical points, not all walls of the diagram intersect transversally. However generally for IHS with n degrees of freedom, the walls of the bifurcation diagram intersect transversally at the point of the image of the momentum mapping corresponding to a nondegenerate singularity.

For more convenient work we will divide the right critical points into several classes.

Table 3. Название таблицы

	$(+,+,+)$	$(-,+,+)$	$(-,-,+)$	$(+,-,+)$	$(+,+,-)$	$(-,+,-)$	$(-,-,-)$	$(+,-,-)$
$(>,>,>)$	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8
$(>,>,<)$	v_4	v_3	v_2	v_1	v_8	v_7	v_6	v_5
$(<,>,>)$	v_5	v_6	v_7	v_8	v_1	v_2	v_3	v_4
$(<,>,<)$	v_8	v_7	v_6	v_5	v_4	v_3	v_2	v_1
$(>,<,>)$	v_5	v_6	v_7	v_8	v_1	v_2	v_3	v_4
$(>,<,<)$	v_8	v_7	v_6	v_5	v_4	v_3	v_2	v_1
$(<,<,>)$	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8
$(<,<,<)$	v_4	v_3	v_2	v_1	v_8	v_7	v_6	v_5

Definition 6. *A right critical point $P = (\xi_1, \xi_2, \xi_3)$ will be called boundary if it lies on the boundary of the momentum mapping image. Otherwise, P will be called internal.*

Definition 7. *Let us call the number of pairs of identical numbers in the set ξ_1, ξ_2, ξ_3, a, b, c by the multiplicity of the critical point $P = (\xi_1, \xi_2, \xi_3)$.*

According to the definition of a proper critical point, its multiplicity does not equal three. Moreover, the following statement is true.

Proposition 4.

1. *When $k > 0$ all right critical points of multiplicity 3 are boundary. These points are $(0, 0, c)$, $(0, 0, b)$, $(0, 0, a)$, and (c, b, a). When $k < 0$ there are exactly three right critical points of multiplicity 3: interior point (c, b, a) and boundary points $(0, c, b)$ and $(0, b, a)$.*

2. *Let P be a right critical point of multiplicity 2. Then,*

   - *when $k > 0$ it lies either on the line $\{\xi_2 = c, \xi_3 = b\}$ and is internal, or lies on the straight lines $\{\xi_1 = 0, \xi_2 = 0\}$ $\{\xi_1 = c, \xi_2 = b\}$, $\{\xi_1 = c, \xi_3 = b\}$, $\{\xi_1 = c, \xi_3 = a\}$, $\{\xi_2 = c, \xi_3 = a\}$, $\{\xi_2 = b, \xi_3 = a\}$, $\{\xi_2 = 0, \xi_3 = c\}$, $\{\xi_2 = 0, \xi_3 = b\}$, $\{\xi_2 = 0, \xi_3 = a\}$, $\{\xi_1 = \xi_2, \xi_3 = c\}$, $\{\xi_1 = \xi_2, \xi_3 = b\}$, $\{\xi_1 = \xi_2, \xi_3 = a\}$, $\{\xi_1 = c, \xi_2 = \xi_3\}$ and is boundary;*
   - *when $k < 0$ it lies either on the lines $\{\xi_1 = c, \xi_2 = b\}$, $\{\xi_1 = c, \xi_3 = a\}$, $\{\xi_2 = b, \xi_3 = a\}$ and is internal, or lies on the lines $\{\xi_1 = b, \xi_2 = a\}$, $\{\xi_1 = c, \xi_2 = a\}$, $\{\xi_1 = 0, \xi_2 = a\}$, $\{\xi_1 = 0, \xi_2 = b\}$, $\{\xi_1 = 0, \xi_2 = c\}$, $\{\xi_1 = b, \xi_3 = a\}$, $\{\xi_1 = c, \xi_3 = b\}$, $\{\xi_1 = 0, \xi_3 = b\}$, $\{\xi_1 = 0, \xi_3 = a\}$, $\{\xi_2 = c, \xi_3 = b\}$, $\{\xi_2 = c, \xi_3 = a\}$, $\{\xi_1 = \xi_2, \xi_3 = b\}$,$\{\xi_1 = \xi_2, \xi_3 = a\}$, $\{\xi_1 = c, \xi_2 = \xi_3\}$, $\{\xi_1 = b, \xi_2 = \xi_3\}$, $\{\xi_1 = 0, \xi_2 = \xi_3\}$ and is boundary.*

Proof. It follows from the structure of the bifurcation diagrams (see Fig. 4) and the definition of a right critical point. □

4. REGULAR POINTS OF THE MOMENTUM MAPPING IMAGE AND THE CORRESPONDING LIOUVILLE FOLIATION LAYERS

Theorem 1. *Let $P = (\xi_1, \xi_2, \xi_3)$ be a regular point of the momentum mapping image of the billiard in a Hooke potential field inside an ellipsoid and T_P be an integral level surface in M^6 corresponding to the point P. Then, the surface T_P is homeomorphic to one or a disjoint union of several three-dimensional tori. The Liouville foliation in a small neighborhood of T_P is trivial.*

Proof. We will number the chambers of the bifurcation diagrams by the numbers of the equivalence classes of the regions of possible motion corresponding to them. In the case $k > 0$ the chambers are numbered with Roman numerals from I to IV, and in the case $k < 0$ from I to VIII (see tables 1, 2 and Figs. 2 and 3).

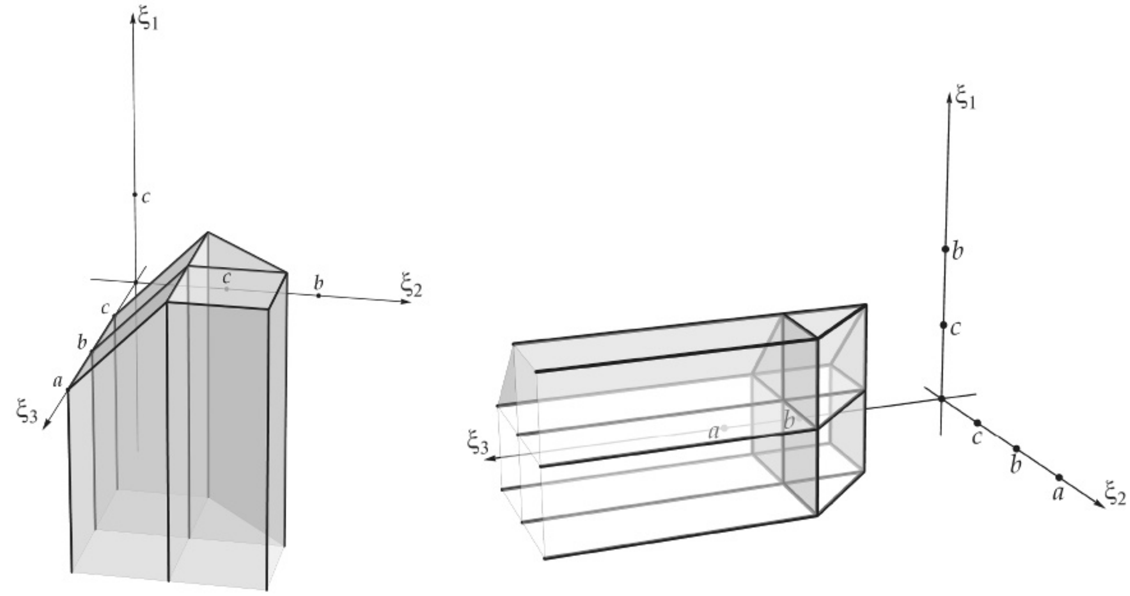

Fig. 4. Bifurcation diagrams of the billiards bounded by the ellipsoid $\mathcal{E}$ (i.e., the quadric of parameter 0) in a Hooke potential field. The diagram for the attractive potential ($k > 0$) is on the left, the diagram for the repulsive potential ($k < 0$) is on the right.

Let us consider the case of an attracting potential and chamber I of the corresponding bifurcation diagram. The other cases are analysed by analogy. Let $P = (\xi_1, \xi_2, \xi_3) \in \mathrm{I}$. Let us denote by D_P the corresponding RPM. At each point $p \in D_P$ consider all vectors v such that $(p, v) \in T_P$. According to the equations of motion 2, exactly 8 such vectors arise at each interior point of D_P that does not lie in any of the coordinate planes. Indeed, at these points, the elliptic coordinates satisfy the restrictions $\lambda_1 \in (\max\{\xi_1, 0\}, \xi_2)$, $\lambda_2 \in (c, \xi_3)$, $\lambda_3 \in (b, a)$, and, hence, $\lambda_i \neq \lambda_j$ when $i \neq j$ and $V(\lambda_i) > 0$. Let's find out how many vectors are located at the remaining points of the RPM.

Let us consider interior points of D_P lying in the coordinate planes. Consider all interior points located in the plane $x = 0$ but not lying in the planes $y = 0$, $z = 0$. Since the plane $x = 0$ in the elliptic coordinates is given by the equation $\lambda_3 = a$, the third elliptic coordinate degenerates at its points. At the same time, the other elliptic coordinates are not degenerate and the values $\left|\dot{\lambda}_1\right|$, $\left|\dot{\lambda}_2\right|$ are not zero. The pair of coordinates (λ_1, λ_2) in the plane $x = 0$ when $y, z \neq 0$ is a smooth local coordinate system. Hence, if we project the velocity vectors at the considered points onto tangent plane $T_{(0,y,z)}Oyz$, we obtain 4 vectors in the projection. Let us show that $|\dot{x}| \neq 0$. For this purpose, we note that in the region of nondegeneracy of the elliptic coordinates the following formula is true

$$\dot{x} = \frac{\partial x}{\partial \lambda_1}\dot{\lambda}_1 + \frac{\partial x}{\partial \lambda_2}\dot{\lambda}_2 + \frac{\partial x}{\partial \lambda_3}\dot{\lambda}_3.$$

If λ_3 tends to a, the derivatives of $\dfrac{\partial x}{\partial \lambda_1}$ and $\dfrac{\partial x}{\partial \lambda_2}$ will tend to zero. So, in the plane $x = 0$, the equality $\dot{x}^2 = \lim\limits_{x \to a} \left(\dfrac{\partial x}{\partial \lambda_3}\dot{\lambda}_3\right)^2$ is true. Using the equations of motion and the elliptic coordinate transition formulae, we obtain

$$\dot{x}^2 = \frac{k\,(a - \xi_1)\,(a - \xi_2)\,(a - \xi_3)}{(a - \lambda_1)\,(a - \lambda_2)}.$$

Since we consider the case $a > \xi_3$, we conclude that $|\dot{x}| \neq 0$. And, hence, at the points of D_P lying in the plane $x = 0$, but not lying in other coordinate planes there are again 8 velocity vectors. The remaining types of interior points of D_P are analyzed similarly.

To each smooth face of the boundary of the region D_P there correspond exactly 4 pairs of nonequivalent vectors, to the smooth edges of the boundary exactly 2 pairs.

Using the location of the velocity vectors at the points of the RPM, let us describe the topology of the layer T_P. For this purpose, we introduce the notation of velocity vectors at internal points of D_P that do not lie in the planes $x = 0$, $y = 0$, and $z = 0$. These planes separate D_P into 8 connectivity components. Each of them is defined by three inequalities: $x > 0$ (or $x < 0$), $y > 0$ (or $y < 0$), and $z > 0$ (or $z < 0$). Let us associate to the vector $(>, >, <)$ the part of D_P that lies in the coordinate octant $x > 0$, $y > 0$, $z < 0$, etc. Consider an interior point of D_P that does not lie in the planes $x = 0$, $y = 0$, and $z = 0$. The velocity vectors corresponding to this point can be uniquely encoded by triples of component signs in the elliptic coordinates $\left(\text{sign}\,\dot{\lambda}_1, \text{sign}\,\dot{\lambda}_2, \text{sign}\,\dot{\lambda}_3\right)$. Using the encodings of the RPM's sub-areas and velocity vectors, let us fill the table of notations.

Note that the same set of signs corresponds to different notations of vectors. Nevertheless, due to this numbering, the vector fields v_i can be extended to the whole set D_P by continuity. Moreover, the vector fields v_i will be smooth inside D_P.

The vector fields $v_1, \ldots, v_8$ divide the surface T_P into 8 components, each of which is homeomorphic to D_P. We denote these components by $D_1, \ldots, D_8$, respectively. Due to the billiard reflection as well as the structure of D_P, we conclude that D_1 and D_2, D_3 and D_4, D_5 and D_6, D_7 and D_8 identify on elliptic boundaries, and D_1 and D_4, D_2 and D_3, D_5 and D_7, D_6 and D_8 identify on hyperbolic boundaries. By gluing D_{2k-1} with D_{2k} $(k = 1, \ldots, 4)$ on elliptic boundaries, we obtain a region homeomorphic to the direct product of two-dimensional torus and a segment. By gluing the obtained regions along the corresponding hyperbolic boundaries, we obtain two three-dimensional tori T^3, which will be formed by the components $D_1, \ldots, D_4$ and $D_5, \ldots, D_8$. Thus, we have proved the first part of the theorem.

It remains to note that at small change of the point P the region of possible motion D_P will not change its type, and the vector fields v_i will change continuously. So, near T_P the Liouville foliation is trivial. □

Below we give a table showing the number of Liouville tori corresponding to points of the bifurcation diagram chambers.

5. LIOUVILLE FOLIATION NEAR CRITICAL LAYERS

In this paragraph, we will describe the Liouville foliation of our billiard in the neighborhood of the layers corresponding to the right critical points (see Definition 5). Before that, however, we recall the notion of a 2-atom introduced by A.T. Fomenko (see, e.g., [23]).

Definition 8. *Let c be a critical value of a Morse function f on a compact orientable manifold M^2. Let $\varepsilon > 0$ be chosen such that on the segment $[c - \varepsilon, c + \varepsilon]$ the point c is the unique critical value of f. The connected component of the set $f^{-1}([c - \varepsilon, c + \varepsilon])$ stratified on the level lines of the function f is called 2-atom.*

Remark 4. All 2-atoms are considered with respect to a layer-by-layer diffeomorphism.

Let us give some examples of 2-atoms. According to Morse's lemma, in a neighborhood of a nondegenerate critical point P, the function f is reduced to the form $f = f(P) \pm x^2 \pm y^2$. Therefore, if P is a point of minimum or maximum, the 2-atom corresponding to it is a circle stratified by the family of concentric circles (see Fig. 5.1). Such an atom is called the *2-atom A*. There are infinitely many atoms corresponding to saddle singularities. In the present paper we will need only two of them: B and C_2. They are shown in Figs. 5.2 and 5.3, respectively. Note that the atoms B and C_2 are centrally symmetric. Actually, the atom C_2 has more symmetries. Indeed, let us apply the inversion mapping to the picture 5.3. The atom C_2 maps into itself. We will also call this involution the central symmetry. To distinguish these two involutions on C_2, we will call them first and second central symmetry. These two symmetries are equivalent.

According to the results of N.T. Zung, the Liouville foliation in a neighborhood of many saddle nondegenerate singularities can be represented as an almost direct product of 2-atoms (see, e.g., [24]). For arbitrary nondegenerate singularities of complexity 1 of systems with 2 degrees of freedom, the realization of their topological invariants in the class of billiard books in a Hooke potential field was done in the paper by V.A. Kibkalo and A.T. Fomenko [28]. The realization of arbitrary semi-local focal singularities of a system with 2 degrees of freedom was also obtained in [29]. It turns out that a similar

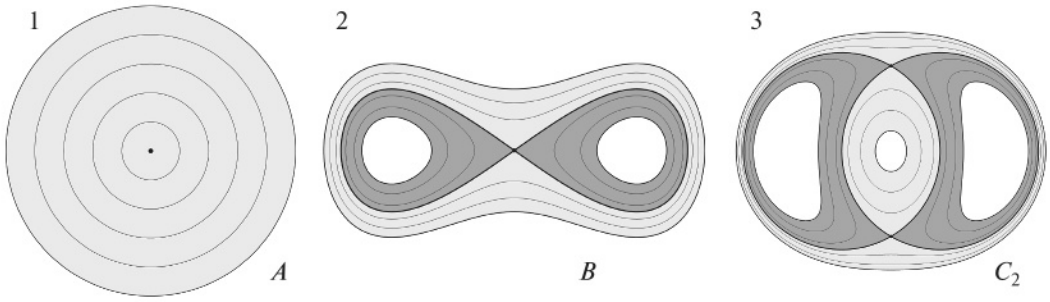

Fig. 5. Examples of 2-atoms (from left to right): A, B, C_2.

representation is valid for the singularities corresponding to right critical points of a billiard in a Hooke potential field inside an ellipsoid.

The higher the multiplicity of a critical point, the more complicated the Liouville foliation in a neighborhood of the layer corresponding to it. According to the Proposition 4, the bifurcation diagram of a billiard in an attractive potential field contains exactly one line of interior critical points of multiplicity 2 (the line $\{\xi_2 = c, \xi_3 = b\}$) and does not include interior points of multiplicity 3. In the repulsive case there is exactly one interior critical point of multiplicity 3 (the point (c, b, a)). If we describe the structure of the Liouville foliation in neighborhood of the layers corresponding to these points, we uniquely identify which bifurcations correspond to all the remaining interior critical points.

Theorem 2.

1. *If $k > 0$, then a small neighborhood of the Liouville foliation layer corresponding to an interior critical point of multiplicity 2 is layer-by-layer homeomorphic to the almost direct product $\dfrac{B \times C_2}{\mathbb{Z}_2(\alpha)} \times S^1 \times \bar{D}^1$, where $\bar{D}^1$ is a segment, and the involution α acts on each of the 2-atoms by the central symmetry.*

2. @ *If $k < 0$, then a small neighborhood of the Liouville foliation layer corresponding to the point (c, b, a) (i.e., the interior critical point of multiplicity 3) is layer-by-layer homeomorphic to the almost direct product $\dfrac{B \times C_2 \times C_2}{\mathbb{Z}_2(\alpha) \times \mathbb{Z}_2(\beta)}$, where the involution α acts by the central symmetry on B and by the first central symmetry on the first C_2 (on the second C_2 acts trivially), and the involution β acts by the second central symmetry on the first C_2 and by the central symmetry on the second C_2 (on B acts trivially).*

Proof. 1. Let $P = (\xi_1^0, c, b)$. Let us denote by T_P the inverse image of the point P in M^6 under the momentum mapping $\mathcal{F}$. The critical circle $\gamma(P)$ arising at motion of the particle along the axis Ox corresponds to the point P. It turns out that on Liouville tori $T_{P'}$ close to T_P one can choose a cycle $\gamma(P')$ homologous to $\gamma(P)$, which, when P' tends to P, will pass to $\gamma(P)$. This is clearly shown in Fig. 6 for all four kinds of Liouville tori close to the surface T_P.

Next, we analyze in detail the case $\xi_1^0 < 0$. For $\xi_1^0 \geqslant 0$ the idea of the proof remains the same, since the reflection from the boundary ellipsoid $\mathcal{E}$ is replaced by the tangent of the ellipsoid of the parameter ξ_1^0. From the point of view of the Liouville foliation topology, the qualitative situation will not change under this transformation. Let us denote by U_P a small neighborhood of the layer T_P.

Note that the forms $\alpha = p_1 d\lambda_1 + p_2 d\lambda_2 + p_3 d\lambda_3$ and $\omega = dp_1 \wedge d\lambda_1 + dp_2 \wedge d\lambda_2 + dp_3 \wedge d\lambda_3$ are correctly defined on the billiard phase space (when $\xi_1^0 > 0$ or $\xi_1^0 < 0$). This follows from the results of V.F. Lazutkin [30] and E.A. Kudryavtseva [31]. Let us define the function s in the neighborhood U_P on the regular parts of the Liouville foliation by the following formula

$$s(P') = \frac{1}{2\pi} \int\limits_{\gamma(P')} \alpha.$$

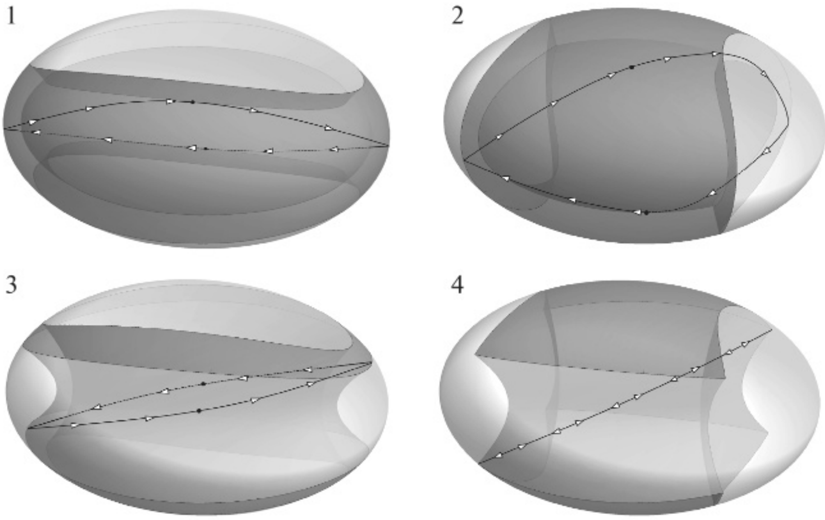

Fig. 6. The cycle γ on Liouville tori close to the isointegral surface $T_P = \mathcal{F}^{-1}(\xi_1^0, c, b)$. The selected points are the tangency points of the cycle γ and the ellipsoid (cases 1, 2), the one-sheeted hyperboloid (case 3) of smaller parameters.

This function will help us construct a trivial S^1-fibration of the neighborhood U_P.

Let us find the explicit form of the function s. According to the formulae of separation of variables (see the Proposition 2), the following equations are true

$$p_i^2 = \frac{k}{4}\frac{(\lambda_i-\xi_1)(\lambda_i-\xi_2)(\lambda_i-\xi_3)}{(a-\lambda_i)(b-\lambda_i)(c-\lambda_i)} \quad \forall i = 1,2,3.$$

The cycle $\gamma(P')$ rounds each elliptic coordinate exactly 4 times, hence,

$$s(\xi_1,\xi_2,\xi_3) = \frac{1}{\pi}\int\limits_0^{\min\{\xi_2,c\}} \sqrt{k\frac{(t-\xi_1)(t-\xi_2)(t-\xi_3)}{(a-t)(b-t)(c-t)}}dt$$

$$+\frac{1}{\pi}\int\limits_{\max\{c,\xi_2\}}^{\min\{\xi_3,b\}} \sqrt{k\frac{(t-\xi_1)(t-\xi_2)(t-\xi_3)}{(a-t)(b-t)(c-t)}}dt + \frac{1}{\pi}\int\limits_{\max\{b,\xi_3\}}^{a} \sqrt{k\frac{(t-\xi_1)(t-\xi_2)(t-\xi_3)}{(a-t)(b-t)(c-t)}}dt.$$

Lemma 1. *The function $s(\xi_1,\xi_2,\xi_3)$ is analytic in a small neighborhood of the point P.*

Proof. Let $\xi_2 \neq c$, $\xi_3 \neq b$. Consider in the complex plane the contour C shown in the Fig. 7. It consists of the upper semicircle L connecting the points 0 and a, four upper semicircles L_ε of radius ε with centers at the points c, b, ξ_2, ξ_3 and five segments. Let's choose a positive direction on the contour. Let

$$f(z) = \frac{1}{\pi}\sqrt{\frac{k(z-\xi_1)(z-\xi_2)(z-\xi_3)}{(a-z)(b-z)(c-z)}}.$$

According to the Cauchy integral theorem,

$$0 = \oint\limits_{C^+} f(z)dz = \int\limits_{L^+} f(z)dz + I_1(\varepsilon) + I_2(\varepsilon) + I_3(\varepsilon),$$

where

$$I_1(\varepsilon) = \int\limits_0^{\min\{\xi_2,c\}-\varepsilon} f(z)dz + \int\limits_{\max\{c,\xi_2\}+\varepsilon}^{\min\{\xi_3,b\}-\varepsilon} f(z)dz + \int\limits_{\max\{b,\xi_3\}-\varepsilon}^{a} f(z)dz,$$

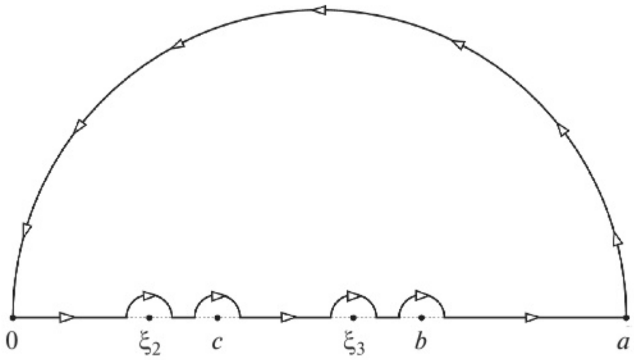

Fig. 7. Contour of integration.

$$I_2(\varepsilon) = \int\limits_{\min\{\xi_2,c\}+\varepsilon}^{\max\{\xi_2,c\}-\varepsilon} f(z)dz + \int\limits_{\min\{b,\xi_3\}+\varepsilon}^{\max\{\xi_3,b\}-\varepsilon} f(z)dz,$$

$$I_3(\varepsilon) = \int\limits_{L_\varepsilon^-(\xi_2)} f(z)dz + \int\limits_{L_\varepsilon^-(\xi_3)} f(z)dz + \int\limits_{L_\varepsilon^-(c)} f(z)dz + \int\limits_{L_\varepsilon^-(b)} f(z)dz.$$

Note that $s = \lim\limits_{\varepsilon\to 0} I_1(\varepsilon)$, the integral $I_2(\varepsilon)$ is imaginary, and $\lim\limits_{\varepsilon\to 0} I_3(\varepsilon) = 0$. Indeed, the latter fact follows from the classical inequality

$$\left|\oint\limits_{\gamma} f(z)dz\right| \leqslant \max_{z\in\gamma} |f(z)|\,|\gamma|.$$

Thus,

$$s = \frac{1}{\pi}\,\mathsf{Re}\int\limits_{L^-}\sqrt{\frac{k(z-\xi_1)(z-\xi_2)(z-\xi_3)}{(a-z)(b-z)(c-z)}}dz.$$

Since there are no critical points c, b, ξ_2, ξ_3 on the contour L, the function s is correctly defined and analytic in a small neighborhood of the point P. The lemma is proved. □

Note that the function s is an action variable.

Using methods of integration of functions of a complex variable, it can be shown that the vector field $v = \mathrm{sgrad}s$ does not equal zero in a small neighborhood of the layer T_P. Moreover, this field is transversal to the submanifold given by the equation $x = 0$ in U_P.

According to the classical theory of IHS, the trajectories of the field v are closed and the integral curves are 2π-periodic. Moreover, the trajectories of v on the layer $T_{P'}$ are homologous to cycles $\gamma(P')$, which are homologous to $\gamma(P)$ by their definition. Now, using the vector field v, we describe the Liouville foliation in the neighborhood U_P.

We denote by W_ε the small cubic neighborhood of the point P, i.e., $W_\varepsilon = [\xi_1^0 - \varepsilon, \xi_1^0 + \varepsilon] \times [c-\varepsilon, c+\varepsilon] \times [b-\varepsilon, b+\varepsilon]$. Without loss of generality, we can assume that $U_P = \mathcal{F}^{-1}(W_\varepsilon)$. Let us represent W_ε in the following form

$$W_\varepsilon = \bigcup_{q\in[-\varepsilon,\varepsilon]} W_\varepsilon(q), \quad \text{where} \quad W_\varepsilon(q) = \{\xi_1^0 + q\} \times [c-\varepsilon, c+\varepsilon] \times [b-\varepsilon, b+\varepsilon].$$

Let us consider the part of U_P that is given by the conditions: $x = 0$, $\dot{x} > 0$. Let us denote it by $\hat{M}$. For arbitrary $q \in [-\varepsilon, \varepsilon]$ we restrict $\mathcal{F}^{-1}(W_\varepsilon(q))$ to $\hat{M}$. This restriction $\tilde{M}$ will be a four-dimensional submanifold in M^6 with boundary. Consider the Cauchy problem of the vector field v with initial points on the manifold $\tilde{M}$. We obtain the flow g_t under the action of which the manifold $\tilde{M}$ deforms inside

$\hat{M}$. However, note that $g_{2\pi} = g_0 = \mathrm{id}$. In other words, in time $t = 2\pi$ the manifold $\tilde{M}$ returns to its initial position. Thus, we obtain the mapping $G : \tilde{M} \times S^1 \to \hat{M}$, which acts according to the formula $G(x, t) = g_t(x)$. Let us list the properties of this mapping.

- The mapping G is continuous. This fact follows from the theorem on the continuous dependence of a solution of the Cauchy problem on an initial data.

- The mapping G is bijective. Indeed, the manifold $\tilde{M}$ intersects every Liouville torus $T_{P'}$ by a two-dimensional torus $\tilde{T}_{P'}$. In this case, the vector field $v = \mathrm{sgrad} s$ will be transversal to $\tilde{T}_{P'}$. It means that the integral trajectories of the field v drawn from different points of $\tilde{T}_{P'}$ will not intersect. Hence, we conclude the injectivity of the mapping G on each of the tori $\tilde{T}_{P'} \times S^1$. If we complete an arbitrary basis on this two-dimensional torus and add to it the cycle $\gamma(P')$, the resulting triple will be a basis in the group of first homologies of the torus $T_{P'}$. Therefore, the whole Liouville torus $T_{P'}$ is covered by the mapping G. Hence one can show the bijectivity of G on the whole product $\hat{M} \times S^1$

Since $\tilde{M}$ and S^1 are compact and the mapping G is a continuous bijection, G is a homeomorphism.

We restrict the Liouville foliation in M^6 to $\tilde{M}$ and extend it to $\tilde{M} \times S^1$, trivially multiplying each layer by the circle. By construction, the mapping G is a layer-by-layer homeomorphism. Therefore, in order to describe the Liouville foliation in the neighborhood U_P of layer T_P, it is necessary to find out how the Liouville foliation in $\tilde{M}$ is structured.

Consider the equations of motion 2 induced on $\tilde{M}$. The third elliptic coordinate is fixed on this submanifold: $\lambda_3 = a$. Let us write the equations of motion for the elliptic coordinates (λ_1, λ_2)

$$\dot{\lambda}_1 = \pm\frac{2\sqrt{2}}{(\lambda_2 - \lambda_1)(a - \lambda_1)}\sqrt{\frac{k}{2}(\lambda_1 - q)(\lambda_1 - \xi_2)(\lambda_1 - \xi_3)(a - \lambda_1)(b - \lambda_1)(c - \lambda_1)},$$
$$\dot{\lambda}_2 = \pm\frac{2\sqrt{2}}{(\lambda_1 - \lambda_2)(a - \lambda_2)}\sqrt{\frac{k}{2}(\lambda_2 - q)(\lambda_1 - \xi_2)(\lambda_2 - \xi_3)(a - \lambda_2)(b - \lambda_2)(c - \lambda_2)}.$$

Note that $\xi_1^0 < 0$ and $\lambda_1, \lambda_2 < a$. Hence, the Liouville foliation of our system on $\tilde{M}$ in the neighbourhood of the layer T_P is the same as that of the system defined inside an ellipse with semi-axes b, c and elliptic coordinates (λ_1, λ_2) and given by equations

$$\dot{\lambda}_1 = \pm\frac{2\sqrt{2}}{\lambda_2 - \lambda_1}\sqrt{-\frac{k}{2}(\lambda_1 - \xi_2)(\lambda_1 - \xi_3)(b - \lambda_1)(c - \lambda_1)},$$
$$\dot{\lambda}_2 = \pm\frac{2\sqrt{2}}{\lambda_1 - \lambda_2}\sqrt{-\frac{k}{2}(\lambda_1 - \xi_2)(\lambda_2 - \xi_3)(b - \lambda_2)(c - \lambda_2)}$$

in the neighborhood of the layer $\xi_2 = c, \xi_3 = b$.

The latter system defines the billiard in a Hooke repulsive potential field inside the ellipse with semi-axes b and c. Hence we conclude that $\tilde{M}$ is layer-by-layer homeomorphic to a small neighborhood of the critical layer of the saddle point of rank 0 of this planar billiard. According to the results of V.A. Kibkalo and A.T. Fomenko (see [28]) this neighborhood is layer-by-layer homeomorphic to the almost direct product of 2-atoms B and C_2 with factorisation by the involution of the central symmetry of these atoms. Thus $\hat{M} \cong (B \times C_2)/\mathbb{Z}_2 \times S^1$.

Since all arguments are true for all $q \in [-\varepsilon, \varepsilon]$ and any fibration over a segment is trivial, the small neighborhood of layer T_P is layer-by-layer homeomorphic to the almost direct product $(B \times C_2)/\mathbb{Z}_2(\alpha) \times S^1 \times \bar{D}^1$, where α is the involution of central symmetry. Thus, we have proved the first statement of the theorem.

Before starting the proving of the second statement of the theorem, we note that the flow g_t at time $t = \pi$ in some sense flips $\tilde{M}$. Namely, a point p with a velocity vector v passes to the point $-p$ with velocity vector $-v$ after time π. In other words, if we considered the billiard not inside the ellipsoid, but in its half cut off by the plane $x = 0$, the small neighborhood of the layer corresponding to the interior point of multiplicity 2 when $k > 0$ would be homeomorphic to the almost direct product $\dfrac{B \times C_2 \times S^1}{\mathbb{Z}_2(\alpha) \times \mathbb{Z}_2(\beta)} \times \bar{D}^1$. Here the involution α acts by central symmetry on the atoms B and the fist C_2 (trivially acting on the circle), and β acts by the additional central symmetry on the first C_2 and S^1 (trivially acting on B). This is indeed true, since inside the half ellipsoid the flow g_t in time π transforms the submanifold $\tilde{M}$ into itself with a twist: the pair (p, v) transforms into the pair $(-p, -v)$, which corresponds to the central symmetry of the almost direct product $\dfrac{B \times C_2}{\mathbb{Z}_2(\alpha)}$. Using this consideration, we prove the second statement of the theorem.

2. Let $P = (c, b, a)$, T_P be the corresponding Liouville foliation layer and W_ε be a small cubic neighborhood of the point P. We show that the neighborhood $\mathcal{F}^{-1}(W_\varepsilon)$ of layer T_P in M^6 is layer-by-layer homeomorphic to the almost direct product $\dfrac{B \times C_2 \times C_2}{\mathbb{Z}_2(\alpha) \times \mathbb{Z}_2(\beta)}$, where α is the central symmetry on B and the first C_2, and β is the additional central symmetry on the first and the second C_2.

Let us represent the neighborhood W_ε in the following form

$$W_\varepsilon = \bigcup_{q \in [-\varepsilon, \varepsilon]} W_\varepsilon(q), \text{ where } W_\varepsilon(q) = [c - \varepsilon, c + \varepsilon] \times [b - \varepsilon, b + \varepsilon] \times \{a - q\}.$$

For all $q \in [-\varepsilon, \varepsilon]$ we describe the structure of the Liouville foliation in $\mathcal{F}^{-1}(W_\varepsilon(q))$.

Let $q > 0$. In this case, we get into chambers V–VIII of the corresponding bifurcation diagram. The regions of possible motion corresponding to these chambers coincide with the regions of possible motion I–IV of the billiard with an attractive potential. Moreover, $\xi_3 = a + q > \lambda_i$ for all i. This means that we can remove multipliers of the form $\xi_3 - \lambda_i$ in the equations with separated variables (see the proposition 2) without changing the topology of the Liouville foliation. As a result, we obtain the system of equations which will be similar to the equations of motion of a material point in the Hooke's attractive potential field. Applying the method described above, we conclude that for $q > 0$ the subset $\mathcal{F}^{-1}(W_\varepsilon(q))$ is layer-by-layer homeomorphic to the almost direct product $\dfrac{B \times C_2}{\mathbb{Z}_2(\alpha)} \times S^1$, where α is the involution of central symmetry.

When $q < 0$ we can take a similar approach. However, we cannot simply remove the multipliers $\xi_3 - \lambda_i$ because they effect types of region of possible motion. We need to change the billiard table. We need to go from a region inside the ellipsoid to two sub-regions given by the inequality $\lambda_3 \leqslant \xi_3$. In each of these regions, the modified equations of motion (i.e., without multipliers $\xi_3 - \lambda_i$) will be almost the same as for a billiard with an attractive potential. Therefore, in view of the remark after the proof of the first statement of the theorem, the subset $\mathcal{F}^{-1}(W_\varepsilon(q))$ in M^6 is homeomorphic to two almost direct products $\dfrac{B \times C_2 \times S^1}{\mathbb{Z}_2(\alpha) \times \mathbb{Z}_2(\beta)}$, where the involution α acts by central symmetry on the atoms B and C_2 (trivially acting on the circle), and β acts by the additional central symmetry on the second C_2 and S^1 (trivially acting on B), when $q < 0$.

Note that when $q = 0$, the two complexes $\dfrac{B \times C_2 \times S^1}{\mathbb{Z}_2(\alpha) \times \mathbb{Z}_2(\beta)}$ are glued on a layer $\dfrac{B \times C_2}{\mathbb{Z}_2(\alpha)}$. This gluing can be represented as the almost direct product of $\dfrac{B \times C \times K^1}{\mathbb{Z}_2(\alpha) \times \mathbb{Z}_2(\beta)}$, where K^1 is the critical layer of C_2, α is the central symmetry on B and C_2, β is the additional central symmetry on C_2 and K^1.

Studying the evolution of $\mathcal{F}^{-1}(W_\varepsilon(q))$, we conclude that the neighborhood U_P of layer T_P is indeed layer-by-layer homeomorphic to the almost direct product $\dfrac{B \times C_2 \times C_2}{\mathbb{Z}_2(\alpha) \times \mathbb{Z}_2(\beta)}$, where α is the central symmetry on B and the first C_2, and β is the additional central symmetry on the first and the second C_2. □

As already mentioned, using the result of the Theorem 2, we can define bifurcations corresponding to all interior critical points. Below we give two theorems that describe the Liouville foliation in the

Table 4. Number of Liouville tori corresponding to the chambers of the bifurcation diagrams.

$k>0$		$k<0$			
I	2	**I**	2	**V**	2
II	2	**II**	4	**VI**	2
III	2	**III**	2	**VII**	2
IV	1	**IV**	2	**VIII**	1

Table 5. Liouville foliation in neighborhoods of layers corresponding to interior points of multiplicity 1. Each complex must be directly multiplied by $S^1 \times \bar{D}^2$.

$k>0$		$k<0$					
$\xi_2=c$, $\xi_3<b$	$2\frac{B\times S^1}{\mathbb{Z}_2(\alpha)}$	$\xi_1=c$, $\xi_2<b$, $\xi_3<a$	$2\,\frac{B\times S^1}{\mathbb{Z}_2(\alpha)}$	$\xi_2=b$, $\xi_1<c$, $\xi_3<a$	$2\,B\times S^1$	$\xi_3=a$, $\xi_1<c$, $\xi_2<b$	$C_2\times S^1$
$\xi_2=c$, $\xi_3>b$	$B\times S^1$	$\xi_1=c$, $\xi_2>b$, $\xi_3<a$	$2\,B\times S^1$	$\xi_2=b$, $\xi_1>c$, $\xi_3<a$	$2\,\frac{B\times S^1}{\mathbb{Z}_2(\alpha)}$	$\xi_3=a$, $\xi_1>c$, $\xi_2<b$	$C_2\times S^1$
$\xi_3=b$, $\xi_2<c$	$C_2\times S^1$	$\xi_1=c$, $\xi_2<b$, $\xi_3>a$	$2\,\frac{B\times S^1}{\mathbb{Z}_2(\alpha)}$	$\xi_2=b$, $\xi_1<c$, $\xi_3>a$	$C_2\times S^1$	$\xi_3=a$, $\xi_1<c$, $\xi_2>b$	$2\,B\times S^1$
$\xi_3=b$, $\xi_2<c$	$B\times S^1$	$\xi_1=c$, $\xi_2>b$, $\xi_3>a$	$B\times S^1$	$\xi_2=b$, $\xi_1>c$, $\xi_3>a$	$B\times S^1$	$\xi_3=a$, $\xi_1>c$, $\xi_2>b$	$B\times S^1$

Table 6. Liouville foliation in neighborhoods of layers corresponding to interior points of multiplicity 2 in the case $k<0$. Each complex must be directly multiplied by the segment.

$\xi_2=b,\xi_3=a,\xi_1<c$	$\frac{C_2\times C_2}{\mathbb{Z}_2(\alpha)}\times S^1$	$\xi_2=b,\xi_3=a,\xi_1>c$	$\frac{B\times C_2}{\mathbb{Z}_2(\alpha)}\times S^1$
$\xi_1=c,\xi_3=a,\xi_2<b$	$\frac{B\times S^1}{\mathbb{Z}_2(\alpha)}\times C^2$	$\xi_1=c,\xi_3=a,\xi_2>b$	$B\times B\times S^1$
$\xi_1=c,\xi_2=b,\xi_3<a$	$2\,\frac{B\times C_2\times S^1}{\mathbb{Z}_2(\alpha)\times\mathbb{Z}_2(\beta)}$	$\xi_1=c,\xi_2=b,\xi_3>a$	$\frac{B\times C_2}{\mathbb{Z}_2(\alpha)}\times S^1$

neighborhood of layers corresponding to all internal critical points different from those studied in the theorem 2.

Theorem 3. Let $P=(\xi_1,\xi_2,\xi_3)$ be an interior critical point of multiplicity 1 of the billiard in a Hooke potential field inside an ellipsoid, then the small neighborhood $\mathcal{F}^{-1}(P)$ in M^6 is layer-by-layer homeomorphic to the direct product $V^3\times S^1\times\bar{D}^2$, where V^3 is a 3-complex given in the table 4 for the corresponding set (ξ_1,ξ_2,ξ_3).

Remark 5. In the Table 4, the involution α acts by the central symmetry on multipliers.

Theorem 4. Let $P=(\xi_1,\xi_2,\xi_3)$ be an interior critical point of multiplicity 2 of the billiard in a Hooke's repulsive potential field inside an ellipsoid, then the small neighborhood $\mathcal{F}^{-1}(P)$ in M^6 is layer-by-layer homeomorphic to the direct product $V^5 \times \bar{D}^1$, where V^5 is a 5-complex given by the table 5 for the corresponding set (ξ_1, ξ_2, ξ_3).

Remark 6. In Table 5, the involution α in all cases except $\xi_1 = c$, $\xi_2 = b$, and $\xi_3 < a$ acts by the central symmetry. In the case $\xi_1 = c$, $\xi_2 = b$, and $\xi_3 < a$ the involution α acts by the central symmetry on the atom B and the circle, and β acts by the central symmetry on the atom C_2 and the circle.

It remains to describe the structure of the Liouville foliation near layers corresponding to boundary right points of the bifurcation diagrams.

Theorem 5. *Let $P = (\xi_1, \xi_2, \xi_3)$ be a boundary right point of the bifurcation diagram of the billiard in a Hooke potential field inside an ellipsoid, then the inverse image of the small neighborhood of the point P under the momentum mapping $\mathcal{F}$ in M^6 is layer-by-layer homeomorphic to*

- *one or disjoint union of several direct products of the form $A \times T^2 \times \bar{D}^2$, $A \times A \times S^1 \times \bar{D}^1$, $A \times A \times A$, if the point P does not lie on the boundary of an inner face of the bifurcation diagram. The number of atoms A in all products is equal to the multiplicity of point P, and the number of such products is equal to the number of Liouville tori in the nearest chamber.*
- *a direct product $A \times V^3 \times D^1$ if the point P borders exactly one inner edge of the bifurcation diagram. Here V^3 is a complex from the table 4 corresponding to the point P.*
- *the almost direct product $A \times \dfrac{C_2 \times C_2}{\mathbb{Z}_2(\alpha)}$ if $k < 0$ and $P = (0, b, a)$. Here α is the involution of the central symmetry on the multipliers.*

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CONFLICT OF INTEREST

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