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Billiards with Changing Geometry and Their Connection with the Implementation of the Zhukovsky and Kovalevskaya Cases

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Abstract. The paper presents a class of billiards with varying geometry, the so-called force or evolutionary billiards, which enable us to realize, in the sense of Liouville equivalence, the well-known cases of Zhukovsky and Kovalevskaya for certain energy zones. On the corresponding 4-dimensional open phase submanifolds, the indicated systems are implemented for an increase in energy on all successively occurring isoenergy 3-surfaces.

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1. INTRODUCTION

In recent years, it has been discovered that, using integrable billiards, it is possible to "visually" simulate diverse integrable Hamiltonian systems with two degrees of freedom. In the program work [1] by V.V. Vedyushkina and A.T. Fomenko, a survey of the current state of research data was made, the achieved results were indicated, and a list of topical open questions was given.

1.1. History of the problem

As it turned out, a wide class of topological invariants of the Liouville foliations of integrable systems are realized as topological invariants of Liouville foliations for billiard systems in a suitable class.

The key invariant of this kind is the Fomenko–Zieschang invariant (marked molecule), i.e., a graph with numerical marks classifying the Liouville foliation of the system in restriction to the isoenergy surface Q^3 up to Liouville equivalence (fiberwise homeomorphism). Note also that the removal of numeric marks gives another (a coarser) invariant, the so-called Fomenko invariant (coarse molecule). This invariant classifies the Liouville foliations of integrable systems into nonsingular surfaces Q^3 up to a homeomorphism of their bases. The foundations of the theory of topological classification were laid down in the works of A.T. Fomenko [2, 3, 4] and together with H. Zieschang [5] (for details, see the book by A.V. Bolsinov and A.T. Fomenko [6]).

Note also that Liouville equivalent systems have "identical" closures of the integral trajectories in general position (see [5, 6]). That is, by an appropriate diffeomorphism, "almost all" closures of solutions of the systems under comparison are combined for almost all initial data.

In the works [7, 8, 9] the Fomenko and Fomenko–Zieschang invariants of many integrable systems of dynamics or mathematical physics were realized (implemented) by billiards, and, in [10], billiards were used to realize such invariants of arbitrary geodesic flows on the sphere and torus with an additional integral which is linear or quadratic in momenta.

The problem of classification of billiard tables and integrable billiards on them is closely related to the realization of specific integrable systems by billiards. For example, in the papers [11, 12, 13] *topological billiards* were classified up to structure of domail (table) and up to Liouville equivalence of billiard dynamical systems in these domains.

For a more general class of *billiard books* introduced by V. V. Vedyushkina in [14] and [15], a similar problem remains open. Recall that by a book (a billiard book) one means a two-dimensional CW-complex domain with permutations on its 1-edges. Several flat domains of integrable billiards are glued along the edge of the complex, and the cyclic permutation on this edge defines the transition of the ball from sheet to sheet after hitting the boundary.

At the same time, the already studied subclasses of billiard books made it possible to implement arbitrary "constituent parts" of the Fomenko–Zieschang invariant: its arbitrary Bott 3-atoms (all possible nondegenerate singularities of rank 1), see [14] and [15], and arbitrary numeric marks r, ε, n, see [16, 17, 18]. Note that the implementation result of 3-atoms was successfully developed to the implementation of an arbitrary Liouville foliation base (given by the Fomenko invariant) using a suitable billiard book [19]. Moreover, diverse classes of three-dimensional surfaces (including those that are not Seifert manifolds) were implemented as isoenergy surfaces of integrable billiards on CW-complexes [20].

We also note that this problem is being actively studied for diverse generalizations of flat and piecewise flat integrable billiards: for billiards in the Minkowski metric [21], billiards in potentials field [22], billiards on the surface of confocal quadrics [23], three-dimensional integrable billiards [24], and billiards with slipping [25].

Interesting results also arise when such constructions are combined. For example, the introduction of a potential filed on a piecewise flat table-complex (a book) enables us to go on to the question of implementation of semilocal nondegenerate singularities of rank 0 (equilibria) of integrable systems [26], as well as of various splitting singularities [27]. Another example is the consideration of a billiard on a table with Minkowski metric in a potential field [28].

We also note the study of the noncompactness problem for Liouville fibers and their singularities, which occur both in integrable systems [29, 30, 31] and in billiard systems. Fomenko invariants of systems on unbounded tables were studied by authors, for example, in [11].

1.2. Force Evolutionary Billiards

We note the following general fact in the above-considered systems of billiards without potential field in flat domains and CW-complexes glued from such flat domains.

When a given system is implemented by billiards for different energy levels of the system, we selected their own "implementing billiard." For the classical topological billiards and billiard books (see [14] and [15]), for the energy we can take the length of the velocity vector of a material point (the billiard ball). By choosing the energy value, we obtain a three-dimensional surface which is said to be isoenergy. It is clear that here the energy is just a scale parameter, in the sense that a change in energy does not change the topology of the isoenergy 3-surface (up to homeomorphism).

The following question repeatedly arose: is it possible to discover a new class of billiards that implements the Hamiltonian system "not by parts," but as a whole on the entire phase 4-manifold M^4, i.e., on all successive isoenergy 3-surfaces at once. A. T. Fomenko discovered a new class of billiards, the so-called *force* or *evolutionary* billiards. In them, with a change in the velocity of the ball (the force of the collision with the wall-boundary), both the topology of the billiard table and the reflection law can change. The billiards-states of force billiards depend on a parameter (the energy) and change inside a fixed "billiard support." In [32], we obtained the following results.

A. It turns out that integrable force billiards implement (in the sense of the Liouville equivalence) some important and well-known Hamiltonian systems "entirely," i.e., on the entire phase manifold M^4 at once (possibly except for singular fibers). In other words, the implementation of the system occurs at once on all its regular isoenergy 3-surfaces. As the energy h of the material point increases, the billiard table changes its topology sufficiently "visually," and the reflection-refraction laws on billiard edges can change. Here, step by step, the three-dimensional levels of constant energy of the deforming billiard change. As a result, such a billiard system implements the considered Hamiltonian system (with origins in geometry, topology, or mathematical physics) step by step at all its energy levels. As prime examples, we "entirely" implemented the Euler and Lagrange systems. Also note that, on a suitable interval of energy values, an evolutionary billiard implements the Goryachev–Chaplygin–Sretensky system, which is well known in rigid bodies dynamics (the discovered implementation of this system is not yet complete, and this result is a part of another our work).

B. In the billiard implementation of the Euler case we found "confocal quadrics" as hidden parameters; in the one of the Lagrange case we found "hidden concentric circles." It turns out that a natural deformation of confocal quadrics into circles (arising from the merging of foci) "transforms" the complete set of Liouville foliations of the Euler case into the complete set of Liouville foliations of the Lagrange case. Recall that we integrate the Euler case using quadratic integral, and the Lagrange case using linear integral. Such a "transformation" of a "quadratic" integrable system into a "linearly" integrable one is a very interesting fact. We call such systems *billiard equivalent*.

In this work, we present the force (evolutionary) billiards discovered by us that implement the Zhukovsky and Kovalevskaya systems on the phase 4-dimensional symplectic submanifolds corresponding to some specific energy zones. We underline that each system is implemented (for increasing values of energy) on all successively occurring nonsingular isoenergy 3-surfaces from these 4-dimensional open submanifolds. So far this implementation is partial, i.e., the discovered "zones of implementability" have not yet exhausted all admissible energy values.

2. DEFINITION OF FORCE EVOLUTIONARY BILLIARDS

Let us recall the definitions introduced by A.T. Fomenko (see [32]).

1. The support of a force billiard is a finite connected two-dimensional CW-complex X containing vertices, edges, and two-dimensional domains-sheets L_i homeomorphic to closed simply connected domains in $\mathbb{R}^2$. This complex

is locally flat, i.e., the metric on 2-sheets is Euclidean, and the gluing of two-dimensional sheets along common boundary arcs (edges of the complex, or the spines of the corresponding books) is isometric. Moreover, all angles between intersecting edges are equal to $\pi/2$. A class important for applications is that of billiard books (see [33, 14, 20]).

2. For each value of the energy parameter $H = h$ (from 0 to infinity), consider a closed subcomplex $X(h)$ (not necessarily connected) in the support X. Let us call it a *state* of the force billiard corresponding to h. We assume that X is the union of all states $X(h)$ and that, for $h_1 < h_2$, we have the condition that $X(h_1)$ is contained in $X(h_2)$, i.e., the state $X(h)$ increases as h grows, it "expands." Some wall-edges of the billiard which were previously impenetrable for the ball (the ball was reflected) can become permeable (the ball goes right through to another sheet). Let us call a finite number of energy values $h = 1, 2, \dots, N$ special (singular) values, and the remaining values are said to be regular. While h grows and remains regular, the state $X(h)$ is subject to a homeomorphism. Note that $X = X(\infty) = X(N + \varepsilon)$, i.e., the support coincides with the last state of the billiard.

3. The edges-spines of a state $X(h)$ are arcs of confocal quadrics or segments of focal straight lines. The boundaries of $X(h)$ can be circles degenerating into points at singular h. The support of X and the states $X(h)$ can sometimes be depicted by means of a homeomorphism (not necessarily an isometry) in $\mathbb{R}^3$ as a two-dimensional "model." In Fig. 1, a force billiard and its states detected by us for the Euler case are shown by domains on an ellipsoid in R^3. Fig. 2 similarly shows the force billiard we found for the Lagrange case.

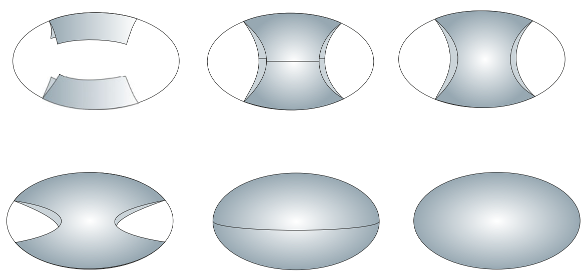

Fig. 1. Example: the locally flat support X of a force billiard simulating Euler's case is homeomorphic to a two-dimensional ellipsoid, and the states $X(h)$ are homeomorphic to smoothly deforming domains on the ellipsoid.

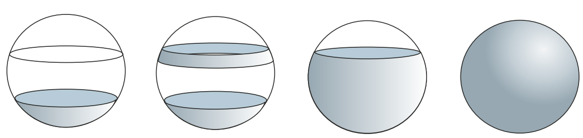

Fig. 2. Example: the locally flat support X of a force billiard simulating the Lagrange case is homeomorphic to a two-dimensional ellipsoid, and the states $X(h)$ are homeomorphic to smoothly deforming domains on a two-dimensional sphere.

4. Denote by $Z(h, r)$ the reflection-refraction law on the edge-spine r in the state $X(h)$. Such edge is equipped by a cyclic permutation from $S(n)$ on the set of n sheets glued by this edge r. Let $Z(h) = \{Z(h, r)\}$ be the family of these laws. We assume that $Z(h)$ is a piecewise constant function and can change only at the singular values $h = 1, \dots, N$. If some edge r becomes "permeable" ("transparent") for one of these h then the billiard ball passes through it for energy levels more than this h.

5. We assume that the edges of the state $X(h)$ can be smoothly varied in the class of confocal quadrics. According to the theory of integrable billiards [34] it specifies equivalent billiards [12]. For critical h, the edges can glue together with other edges, degenerate, turn into segments of focal straight lines. The sheets are glued along an arc of the same quadric that is a boundary arc of these sheets. On the "new spine," (i.e. this edge) a new cyclic permutation appears. At the moment of such state change (a "jump") we allow the billiards to change their equivalence class. For example, a segment of the border with a critical h can lie on the focal line or "fold in half" (see Fig. 1 and Fig. 3).

   By a jump, an angle of $\pi/2$ can become equal to π. It is allowed to glue, at the boundary points, spines of the same state $X(h)$ if they lie on one boundary arc, i.e., when the angle between them becomes equal to π. Boundary circles can contract into points.

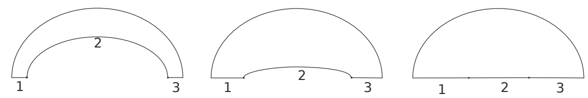

Fig. 3. Combining segments 1, 2, and 3 when “jumping”. As a result of the evolution of the billiard, the segments 1 and 3 passed into segments of the focal line with ends at the foci, and the segment 2 became a segment between foci.

6. Thus, we assume that the enveloping two-dimensional support X is unchanged, “fixed.” The states $X(h)$ “grow” in it, and X coincides with the last state $X(N+\varepsilon)$. An integrable system with two degrees of freedom given by the dynamics of a billiard ball on the changing states $X(h)$ is called a *force* (*evolutionary*) *billiard*. Let h be a regular energy value in some interval $D_i = (i, i+1)$. Denote the corresponding billiard state by $X(D_i)$.

7. A point of the phase complex $TX(D_i)$ is a pair (x, v), where x is a point of the billiard table $X(D_i)$ and v is the velocity vector of a material particle at the point x. If the point x is on the boundary of a sheet L_i adjacent to a sheet L_k, then the corresponding pairs (x, v) and (x, w) are glued according to the reflection-refraction law $Z(h, r)$ acting on the given edge r.

8. By a regular isoenergy 3-surface Q_h, we mean a subset in the four-dimensional phase complex $TX(D_i)$ given by the equation $H = h$; i.e., a constant energy level. For integrable billiard books the regular surfaces Q_h are topological continuous 3-manifolds [33].

Remark 1. • Initially the concept of force billiards included the following requirement. The billiard states should “grow” when the energy increases, i.e., a next state contains (absorb) the one. Here the enveloping support remains unchanged. This implied the requirement that the edges of the states can glue together, but cannot “unglue” (disintegrate). However, as the analysis of specific integrable systems of physics and mechanics showed, it is sometimes useful to weaken the requirements on the force billiard and allow the ungluing of some edges of billiards-states. Thus, we permit the “inverse operation.” Note that here we can keep the condition that all these events develop inside the unchanged, “stationary” support.

- Further, it is possible to additionally weaken the requirement on the force billiard and to refuse the presence of an unchanged enveloping support, replacing it by the condition that the two-dimensional sheets of a previous billiard-state are subsets of two-dimensional sheets of the subsequent one.

- Finally, it makes sense (in some cases) to allow changing the family of confocal quadrics in which the evolution of billiards-states occurs. For example, we will allow to pass from the family of concentric circles to the family of confocal ellipses. In this case, the common center of the circles “splits” into a pair of foci of the family of confocal ellipses (one point splits into two points). This evolution has already been considered by us in the definition of billiard equivalence of integrable systems, see, for example, the Euler and Lagrange cases in the paper [32]. Certainly, one can consider the converse operation (merging of foci into one point).

This more flexible approach expands the possibilities for the implementation of specific systems by force billiards, since it expands the class of force billiards. We will show this on specific examples.

3. INTEGRABLE BILLIARDS BOUNDED BY ARCS OF CONFOCAL QUADRICS

In the paper, we need two classes of integrable billiards, namely, the billiards bounded by arcs of confocal ellipses and hyperbolas and also billiards bounded by arcs of concentric circles and radial straight lines.

We choose a *family of confocal quadrics* by the relation

$$(b-\lambda)x^2 + (a-\lambda)y^2 = (b-\lambda)(a-\lambda).$$

Here $a > b > 0$ are chosen parameters of the family (square roots of the semiaxes of an ellipse with the parameter $\lambda = 0$), which, in particular, determine the distance between the foci. This relation describes a family of confocal ellipses and hyperbolas, which include the focal line $y = 0$ and the limit hyperbola $x = 0$.

By an *elementary billiard* we mean a compact connected part of the plane whose boundary consists of arcs of confocal quadrics and does not contain any angles of $3\pi/2$. Note that confocal quadrics always intersect at right angles. The prohibition of angles $3\pi/2$ enables us to correctly determine the billiard motion after hitting the material points into a corner. Namely, after the reflection, the point continues to move in the opposite direction along the same segment along which it hit the corner. Such billiards are integrable [34]. Every their nonsingular trajectory consists of straight line segments that are tangent to some quadric of the chosen family. Thus, along the trajectory, the parameter λ of this quadric is preserved. In the simplest case (domain bounded by an ellipse) the integrability of the billiard was shown by G. Birkhoff [35].

The singular trajectories here are the trajectories that pass along the convex boundary arcs (the parameter λ on them coincides with the parameter of this boundary), the trajectories that lie on straight lines passing through two foci ($\lambda = b$), and the trajectories passing along the limit hyperbola $x = 0$ (for them, $\lambda = a$). The Fomenko–Zieschang invariants of flat billiards were studied by V. Dragovich and M. Radnovich [36] and by V.V. Vedyushkina [37].

In our paper, we use the class of topological billiards, i.e., two-dimensional orientable manifolds glued from elementary billiards along the arcs of the boundaries in such a way that the billiard reflection is always well defined. In particular, this means that, when gluing the corners of elementary billiards, only two or four elementary billiards can be simultaneously glued together. For details concerning the rules of gluing, see [12].

We also need billiard books, which are CW-complexes glued from elementary billiards along the boundaries. Now more than two elementary billiards are glued along some boundary arcs (the corresponding edges of the complex are called spines). To define a dynamics we enumerate all two-dimensional cells of the resulting complex, i.e. the interiors of the elementary billiards (sheets of a book). After this, to every edge-spine, i.e. a one-dimensional cell of the complex, we assign a cyclic permutation σ consisting of the indices of the elementary billiards (sheets) adjacent to the given 1-cell. A billiard particle moving along a sheet with some index i, after hitting this spine, continues the motion along the sheet with the new index $\sigma(i)$ given by this permutation.

Note that, if all sheets of a topological billiard (billiard book) belong to the same family of confocal quadrics, then the topological billiard (billiard book) thus obtained is also integrable, and with the same pair of integrals, namely, the squared length of the velocity vector and the parameter λ of the confocal quadric.

Billiards whose tables are bounded by arcs of concentric circles and segments of radial lines are also considered. Note that, if in the previously considered family of confocal ellipses and hyperbolas we pass to the limit as the parameter b of the minor semiaxis tends to the parameter a of the major semiaxis, then the confocal ellipses are transformed to concentric circles, and every hyperbola is transformed to a pair of straight lines (its asymptotes). Thus, the class of integrable topological billiards and billiard books whose two-dimensional sheets are bounded by concentric circles and segments of radial straight lines is constructed similarly.

4. ZHUKOVSKY AND KOVALEVSKAYA CASES

The classical integrable tops of rigid body dynamics can be described as dynamical systems on the six-dimensional dual space $e(3)^*$ to the Lie algebra $e(3)$ of the group of motions of three-dimensional Euclidean space. In suitable coordinates $S_1, S_2, S_3, R_1, R_2, R_3$, the Poisson bracket acquires the form

$$\{S_i, S_j\} = \varepsilon_{ijk} S_k, \quad \{R_i, S_j\} = \varepsilon_{ijk} R_k, \quad \{R_i, R_j\} = 0,$$

where $\{i, j, k\} = \{1, 2, 3\}$, and $\varepsilon_{ijk} = \frac{1}{2}(i-j)(j-k)(k-i)$.

A Hamiltonian system on $e(3)^*$ is given by the Euler equations for some Hamiltonian $H \in C^\infty\left(\mathbb{R}^6\right)$:

$$\dot{S}_i = \{S_i, H\}, \quad \dot{R}_i = \{R_i, H\}.$$

Consider the common level surface of two Casimir functions of the given Poisson bracket: the geometric integral f_1 and the area integral f_2,

$$M^4_{c,g} = \left\{f_1 = R_1^2 + R_2^2 + R_3^2 = c, f_2 = S_1 R_1 + S_2 R_2 + S_3 R_3 = g\right\}.$$

For $c > 0$ and any $g \in \mathbb{R}$, the common level surface $M^4_{c,g}$ is a symplectic leaf of the Poisson bracket, i.e., a four-dimensional smooth manifold on which the Poisson bracket is nondegenerate. Below, we assume that c and g are such regular values.

We present below the Hamiltonians and the corresponding additional integrals for two classical integrable systems of rigid body dynamics.

The first system is the case discovered by N.E. Zhukovsky in 1885 [38]. This case describes the motion of Euler's top in the gravity field after adding a gyrostat given by a vector $\lambda = (\lambda_1, \lambda_2, \lambda_3)$:

$$H = \frac{(S_1 + \lambda_1)^2}{2A_1} + \frac{(S_2 + \lambda_2)^2}{2A_2} + \frac{(S_3 + \lambda_3)^2}{2A_3}, \quad K = S_1^2 + S_2^2 + S_3^2.$$

Here, as in the Euler system, the integral K is quadratic in momenta. Euler's case is obtained from Zhukovsky's case when all λ_i's are equal to zero.

Another system for which we shall construct an evolutionary billiard is the famous Kovalevskaya top. It was discovered by S.V. Kovalevskaya in 1886 [39] and is the third top (along with the Euler and Lagrange tops) which is integrable for all possible values of four integrals and initial conditions.

$$H = \frac{S_1^2}{2A} + \frac{S_2^2}{2A} + \frac{S_3^2}{A} + a_1R_1 + a_2R_2,$$
$$K = \left(\frac{S_1^2 - S_2^2}{2A} + a_2R_2 - a_1R_1\right)^2 + \left(\frac{S_1S_2}{A} - a_1R_2 - a_2R_1\right)^2.$$

Here the integral K has degree 4 in momenta.

Remark 2. This system in suitable energy zones (like the Goryachev–Chaplygin–Sretensky system) is Liouville equivalent to integrable billiards on suitable tables [40]. They have the same polynomial billiard integral. Thus, the phenomenon of lowering the degree of the integral was discovered: Liouville foliations of the systems with integrals of degrees 3 and 4 are fiberwise homeomorphic to foliations of billiards with an integral of degree 2 in appropriate energy zones.

Assume that $f_1 = 1$. Then different 3-surfaces Q^3 are defined by the parameters g and h. Consider the bifurcation diagram of the pair of integrals f_2 and H. As a result, curves occur on the plane $R^2(g,h)$ that divide the plane that, for all points (g,h) in some domain, the topological type (the class of homeomorphism) of the corresponding isoenergy surfaces

$$Q^3 = \{f_1 = 1,\ f_2 = g,\ H = h\}$$

is the same. Consider the mapping

$$F = f_2 \times H : S^2 \times R^3 \to R^2(g,h)$$

given by the formula $F(P) = (f_2(P), H(P)) \in R^2(g,h)$.

The image of the set of critical points of the mapping F is the bifurcation diagram in the plane $R^2(g,h)$. It splits the plane into chambers. In every chamber, the preimage of every point is a nonsingular isoenergy surface. The topological type of the 3-manifolds corresponding to different points of the same chamber are the same. In the paper, we need results obtained by M.P. Kharlamov [42] and A.A. Oshemkov [43] for the cases of Zhukovsky and Kovalevskaya. Note that the bifurcation diagrams under consideration are equipped with an additional information. Namely, the chambers of the bifurcation diagram are additionally separated by dashed lines. They correspond to the presence of homeomorphic isoenergy surfaces; in the restriction to these surfaces, the Liouville foliation of the system is not *fiberwise* homeomorphic. These foliations correspond to different types of Fomenko and Fomenko–Zieschang invariants.

5. "PARTIAL" FORCE BILLIARD FOR THE KOVALEVSKAYA CASE

Consider the classical Kovalevskaya top. The bifurcation diagram for the Kovalevskaya case was calculated by M.P. Kharlamov (see [41, 42]. In the papers of A.A. Oshemkov [43] and A.V. Bolsinov [44], the nondegeneracy problem for critical points of rank 1 and 0 was studied, Fomenko invariants of the isoenergy surfaces of the system were also constructed [43], and loop molecules of singular points of the bifurcation diagrams of the momentum mapping (H, K) of the system [44] were found.

The bifurcation diagram on the plane (g,h) of values of the area integral f_2 and the energy H of the system has the form shown in Fig. 4. The solid lines of the diagram separate areas on the plane for which the preimages of their points contain *nonhomeomorphic* isoenergy surfaces. The diagram is also supplemented with dotted curves that separate domains with different types of Liouville foliations on *homeomorphic* isoenergy surfaces. We highlight, in darker color, the chambers of the bifurcation diagram such that the Liouville foliation of the corresponding isoenergy surfaces are currently modelled by integrable billiards.

Consider the symplectic leaf A corresponding to the vertical line $g =$ const. This straight line (see Fig. 4) intersects a five-chambers bifurcation diagram; moreover, for the first three chambers, the Liouville foliation (on the isoenergy surface lying in the inverse image of any point in this chamber) can be modelled by the Liouville foliation of integrable billiards bounded by arcs of confocal quadrics. These billiards, denoted by $\alpha,\ \beta,\ \gamma$, are shown in Fig. 4.

- The elementary billiard α is bounded by an arc of an ellipse (we set its value of the parameter to be $\lambda = 0$), an arc of a nonconvex hyperbola (we set its parameter value to be $\lambda = b + \frac{a-b}{2} = \frac{a+b}{2} < a$), the degenerate hyperbola $x = 0$ (recall that its value of the parameter is $\lambda = a$), and the focal line (recall that its value of the parameter is $\lambda = b$).

- The topological billiard β is glued from two billiards α, where the gluing occurs along both arcs of the hyperbolas.

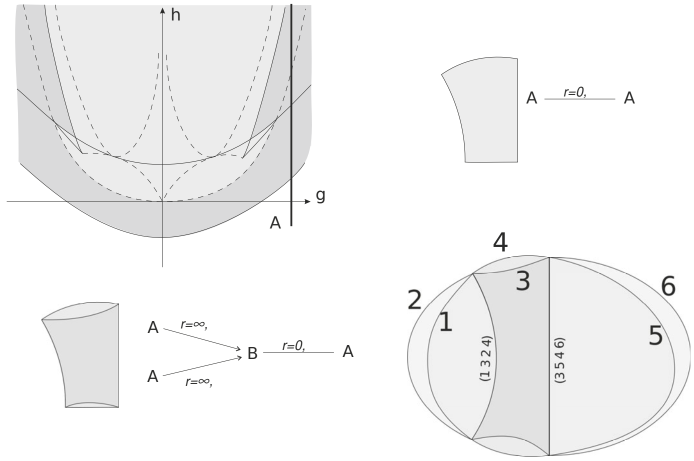

Fig. 4. Bifurcation diagram of the Kovalevskaya case. The dotted line separates the two domains for which Liouville foliations on Q^3 differ, but the isoenergy surfaces Q^3 themselves remain homeomorphic. The domains for which implementing billiards were constructed are highlighted in darker color. The vertical line A corresponds to the symplectic leaf which is partially modelled by the force billiard. By $\alpha,\ \beta,\ \gamma$ we denote parts of this symplectic leaf, as well as force billiard states that simulate (model) these parts. These states are also depicted in Fig. 5.

- The billiard book γ is glued together from six elementary billiards. The billiards with numbers 1 and 2 are bounded by arcs of an ellipse with the parameter $\lambda = 0$ and the hyperbola with the parameter $\lambda = \frac{a+b}{2}$. The billiards with numbers 3 and 4 are also bounded by arcs of an ellipse with the parameter $\lambda = 0$ and hyperbola the with parameter $\lambda = \frac{a+b}{2}$, and also by segments of the degenerate hyperbola with the parameter $\lambda = a$. The billiards with numbers 5 and 6 are bounded by an arc of an ellipse with the parameter $\lambda = 0$ and an arc of the hyperbola with the parameter $\lambda = a$. We assign the following permutations to the hyperbola arcs that are spines of the book; to the hyperbola arc with parameter $\lambda = \frac{a+b}{2}$ we assign the permutation (1 3 2 4) and, to the arc of degenerate hyperbola, the permutation (3 5 4 6).

Note that the billiard α constructed above is a subset of the billiard β, and the billiard β, in turn, is a subset of the billiard γ. This fact enables us to regard them as states of a single force billiard K. For the enclosing billiard of the complex, we take the billiard γ and, for the initial state, we take the billiard α, whose image under the isometric embedding in the billiard book γ covers the top half of the sheet 3. We embed the intermediate position (the billiard β) into the book γ in such a way that the image of this billiard, under the isometric embedding, covers the upper half of the sheets 3 and 4.

Let us now describe the transformations of the constructed force billiard K. At the first jump, at the transformation of the billiard α into the billiard β, a sheet of the billiard is added, and the walls of the hyperbolas become partially permeable. The billiard ball now moves on the sheets 3 and 4 of the billiard book γ, but still cannot overcome the focal line. The second jump adds billiard sheets with numbers 1, 2, 5, and 6, the permutations on the hyperbolic arcs change, and the focal line now becomes permeable.

To prove that the constructed billiard K partially simulates the Kovalevskaya case on the chosen symplectic leaf A indeed, we claim that the Fomenko–Zieschang invariants encoding the Liouville foliation on regular isoenergy surfaces coincide with the corresponding invariants for the billiards α, β, and γ.

For the billiards α and β, the Fomenko–Zieschang invariants (also shown in Fig. 4) were calculated in the work of V.V. Vedyushkina [13]. Let us now calculate the Fomenko–Zieschang invariant for the billiard book γ.

Remark 3. The billiard γ as a billiard book (the set of elementary billiards and permutations on the spines) belongs to the family of billiard books ("butterflies with wings") constructed by V.A. Kibkalo and V.V. Vedyushkina to prove Fomenko's conjecture about the existence of a billiard book whose Fomenko–Zieschang invariant contains an arbitrarily large fixed value of an integer mark n (see [17]). The hyperbolic spines (of the billiard books described in that paper) lie on the arcs of the same hyperbola. In the billiard γ constructed above, one of the arcs lies on a vertical line. As a result, the form of the graph-molecule changes, although the structure of the marks remains the same.

Proposition 1. *The Fomenko–Zieschang invariant encoding the Liouville foliation of an isoenergy surface of the billiard book γ is shown in Fig.* 5 *at the top line to the right.*

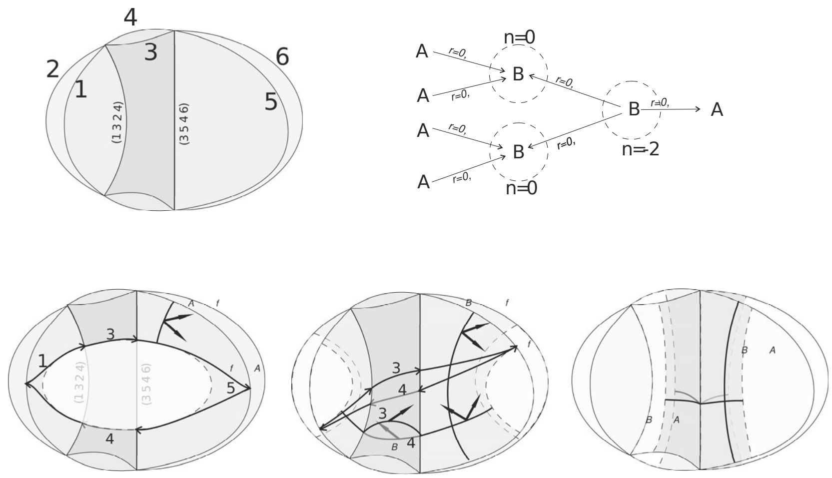

Fig. 5. On the top line: the state γ of the force billiard simulating the Kovalevskaya case, as well as the Fomenko–Zieschang invariant encoding its Liouville foliation. The bottom line shows sequentially the projections, of the cycles on the Liouville tori corresponding to the edges of the molecule going from left to right. By λ_f and μ_f we denote the cycles corresponding to one of the left atoms of B describing the bifurcation at the focal level.

Remark 4. Recall that, in order to calculate the Fomenko–Zieschang invariant encoding the Liouville foliation of a Hamiltonian system, one must take the following three steps. First, construct the Reeb graph of the additional integral of the system on the surface Q^3. The edges of this graph correspond to one-parameter families of regular Liouville tori, and the vertices to their bifurcations. Further, it is necessary to place the so-called atoms at the vertices of the graph. The atoms denoted by letters determine the bifurcations of these Liouville tori. According to Fomenko's theorem [6, Ch. 3, Vol. 1], every such bifurcation possesses a natural Seifert fibration structure whose singular fibers (if exist) are of the type (2,1). The atoms with singular fibers of the Seifert fibration are called atoms with stars. The final step in calculating the invariant consists in adding numerical marks that determine the gluing of the boundary tori of 3-atoms. Choose some edge of the graph. Its ends refer to some atoms. The point on the edge corresponds to the Liouville torus. On this torus, one can choose two cycles, λ and μ, which give a basis in the homology group. The choice of the pair of cycles must be done according to the rules dictated by the atom at the end of the edge (see [5, 6]). The choice of such a pair of cycles is ambiguous. Here the choice of the cycle λ is usually unique up to orientation, and the complementary cycle μ is subjected to a number of significant conditions. Between the selected airs of cycles, we can write out an integer matrix of the transition from one cycle to another (it is called the gluing matrix). Since this matrix glues pieces of the manifold Q^3 which is cut along the Liouville torus, it follows that its determinant is always equal to -1. From the entries of this matrix (which is defined ambiguously, as well as a pair of cycles), the marks r and ε on the edges are evaluated, and the marks n on the subgraphs of the invariant, the so-called families. All the vertices in the families are saddle, and the preimage of the family (i.e., a three-dimensional subset of the manifold Q^3) admits the structure of a Seifert fibration.

Remark 5. When choosing cycles on Liouville tori in billiard systems, we shall proceed as follows. We depict the projection of this cycle on the billiard table. To restore this curve to a cycle on a torus, we must equip the points of this projection by velocity vectors in such a way that every point-vector pair determine a straight line tangent to the quadric fixing this Liouville torus. To this end, we shall act in two ways. We shall supply the projection with arrows that either fix the direction on it (and hence can be naturally continued until equipping the projection with velocity vectors) or explicitly indicate the form of velocity vectors (bold black arrows). As a rule, the first method is convenient for fixing cycles λ on boundary tori of atoms-bifurcations lying at the focal level. On this fiber, the caustics change their type: if the trajectories before the bifurcation are tangent to ellipses, then, after the bifurcation, trajectories are tangent the hyperbolas. In some obvious cases, we shall not indicate the equipping with velocity vectors, since the corresponding cycle is readily reconstructed from its projection to the billiard table.

Proof. Let the parameter λ of the trajectory be strictly less than $\frac{a+b}{2}$. Then all trajectories are divided into two classes. The first class trajectories pass along the sheets with numbers 1 and 5, as well as the sheet 3 (strictly from left to right) and the sheet 4 (strictly from right to left). The trajectories of the other class pass along the sheets with numbers 2 and 4, the sheet 3 strictly from right to left, and the sheet 4 strictly from left to right. Moreover, the trajectories of all classes are in one-to-one correspondence with the billiard trajectories in the ellipse with parameter $\lambda = 0$ whose parameters are also strictly less than $\frac{a+b}{2}$. This enables us to say that the structure of a coarse molecule for tori corresponding to these trajectories can also be taken from the structure of the graph for the billiard in an ellipse. We see that, at the level $\lambda < \frac{a+b}{2}$, there is a bifurcation of four Liouville tori into two tori through two atoms B. At the minimum level, we have four atoms A. Recall that an atom A is a fibration into tori and an axial circle solid torus describing the contraction of tori to the axis of the solid torus, the critical trajectory. The atom B is the direct product of the two-dimensional atom B and the circle. The singular fiber of the two-dimensional atom B is homeomorphic to the figure "eight." Thus, the two-dimensional atom B describes the bifurcation of two circles into one (or one into two), and the three-dimensional atom B describes the bifurcation of two tori into one.

In the bottom line on the left in Fig. 5, cycles are chosen (more precisely, their projections to the billiard domain) on Liouville tori corresponding to trajectories tangent to ellipses. Here the cycles λ_A and μ_A refer to the left atoms A, and the cycles λ_f and μ_f refer to the left atoms B describing the bifurcation on the focal level.

Let us explain the choice of these cycles. The cycle λ_A obviously contracts to a point as the caustic tends to the boundary ellipse, while the cycle μ_A, on one side, complements λ_A to a basis and, on the other hand, passes to a critical motion along an ellipse. The cycles λ_f in the bottom line in Fig. 5 pass into the critical motion along the focal straight line as the parameter of the quadric tends to the value b. Thus, we see that these cycles are homeomorphic to the fibers of the Seifert fibration of the atom B. The cycles μ_f lie on the arcs of some confocal hyperbola transverse to the focal straight line. Being equipped with suitable velocity vectors, they, firstly, are complement the chosen cycles λ_f to a basis and, secondly, form the boundary circles (taken all together) of a two-dimensional atom B which is a two-dimensional section of the 3-atom B transversal to its critical circle (the motion along the focal straight line).

Now let us show that, for $\lambda = \frac{a+b}{2}$, the tori corresponding to the two classes of trajectories described above merge into one torus. Consider some arc of some ellipse in the distinguished confocal family and its projection to the billiard. Let us equip the points of this arc with velocity vectors directed "upwards." In this way we obtain one connected component of the inverse image of this arc for $\lambda \in [\frac{a+b}{2}, a]$. Let us show that this component is homeomorphic to the two-dimensional atom B. For $\lambda < \frac{a+b}{2}$, this part of the preimage consists of two circles (each lies on the Liouville torus of one of the two distinguished classes of trajectories) and, for $\frac{a+b}{2} < \lambda < a$, the connected component of the preimage consists of a single circle. For $\lambda = \frac{a+b}{2}$ two circles merge into one along a point lying on the spine with the parameter $\frac{a+b}{2}$; it is also equipped with a vector directed "upward." The second part of the preimage is obtained if we consider downward directed velocity vectors. The critical circle of the described atom are the points of the hyperbola arc with the parameter $\frac{a+b}{2}$ equipped with tangent velocity vectors. Note that the motion along this arc is not a trajectory. The entire atom has the structure of the direct product of the circle (corresponding to the hyperbola arc equipped with tangent velocity vectors with parameter $\frac{a+b}{2}$) and the two-dimensional atom B.

The choice of the cycles λ_B is motivated by the fact that, as the tori tend to a critical value, these cycles become homeomorphic to the fiber of the Seifert bundle, i.e., to the points of an arc of the hyperbola with the parameter $\frac{a+b}{2}$ equipped with tangent velocity vectors. The cycles μ_B are the cycles considered above that lie in the inverse image of the arc some ellipse in the confocal family.

Before writing out the gluing matrices, let us comment out the choice of orientations on the cycles. The orientations of the cycles λ on saddle atoms B and of the cycles μ on minimax atoms A should be chosen consistent with the orientation of the fibers of the Seifert bundle (i.e., pass to trajectories or to homeomorphic ones with the same orientation). In terms of projections, this implies the following rule: at all interior points, one chooses the direction as the direction of the projection of the velocity vector to the tangent line to the projection-curve. At all other points (for example, at the boundary points), the direction on a cycle can be determined by continuity. This choice of orientation will be called *consistent*. Further, the orientations of the cycles μ on the tori corresponding to saddle atoms B and lying on opposite sides of the critical fiber must be opposite. In particular, we have chosen orientations in such a way that the orientation of the cycle μ_f for $\lambda < b$ will be consistent and, for $\lambda > b$, it is inconsistent. This determines the choice of signs in the relationship $\mu_f = -\lambda_f$, as well as the signs in the gluing matrices written below.

We see that, on the edges between the atoms B and the atoms A, the gluing matrices have the form $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. They define the following marks on the edges: $r = 0,\ \varepsilon = 1$, and give zero contribution to the marks n. On the edges between the atoms B, the gluing matrices have the form $\begin{pmatrix} 1 & -1 \\ -1 & 0 \end{pmatrix}$. We see that the marks on these edges are $r = 0,\ \varepsilon = -1$. The contribution of every edge to the mark n of the family is $[\frac{\alpha}{\beta}] = -1$ if the edge is outgoing for this family. If an edge is incoming in this family, then the contribution is zero. The proposition is proved.

Theorem 1. *The constructed evolutionary billiard K implements* (*in the sense of Liouville equivalence*) *the integrable Kovalevskaya case on the part of the phase symplectic manifold M_g^4 corresponding to the line A in Fig.* 4.

Remark 6. We stress that the evolution of billiard walls occurs in the class of confocal quadrics, which ensures the integrability of the system at every moment of its evolution at all occurring in succession (as the energy grows) isoenergy 3-surfaces.

Proof. We rely on the results of nontrivial calculations made earlier in the works cited below. The proof of the theorem follows from the comparison of the Fomenko–Zieschang invariants for the Kovalevskaya case, calculated by A.V. Bolsinov, P. Richter, and A.T. Fomenko in [45], and the Fomenko–Zieschang invariants of the billiard states α, β (see Fig. 4) and γ (see Fig. 5) of the force billiard K. We omit the complete list of invariants due to large size.

Remark 7. For a complete implementation of the Kovalevskaya system on the entire symplectic 4-manifold corresponding to the line A, it remains to find implementing billiards for the two "upmost" zones in Fig. 4.

Remark 8. Note that the results on the topology of the Kovalevskaya top in [45] were recently extended by V.A. Kibkalo to the case of analogues of the indicated system for the Lie algebra $so(4)$ [46, 47] and the Lie algebra $so(3,1)$ [48]. Here Fomenko–Zieschang invariants not observed earlier were discovered and the cases of equivalence of these systems with other integrable systems were found. In the future, it would be interesting to apply the construction of evolutionary billiards to simulate analogs of Kovalevskaya systems on diverse Lie algebras.

6. "PARTIAL" FORCE BILLIARD FOR THE ZHUKOVSKY CASE

We now turn to the Zhukovsky case. The general form of the bifurcation diagrams for the Zhukovsky case is shown in Fig. 6 a). In Fig. 6 b) and Fig. 6 c) we show two special cases of this bifurcation diagram. The dotted lines separate diverse Liouville foliations of homeomorphic isoenergy surfaces.

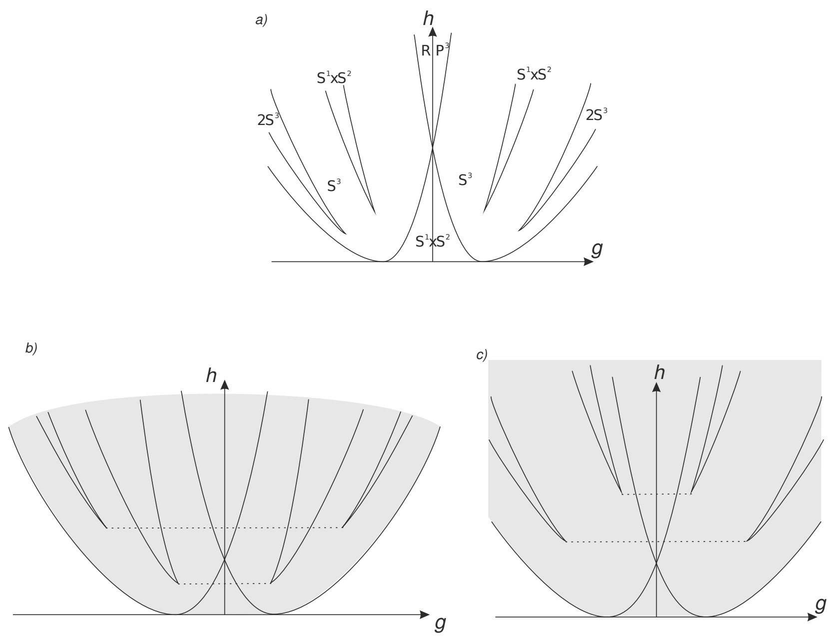

Fig. 6. Bifurcation diagrams of the Zhukovsky case whose arcs separate different types of isoenergy surfaces; *a)* the general case; *b)* and *c)*: two special cases. The dotted lines correspond to the presence of nonequivalent Liouville foliations on homeomorphic isoenergy surfaces.

6.1. Zhukovsky system: the case of the bifurcation diagram b

Let us consider in detail the case b of the bifurcation diagram. Let us mark on it the chambers whose Liouville foliation of isoenergy surfaces which can be modelled by integrable billiards (see Fig. 7). Let us describe these integrable billiards, also marked with the letters α, β, γ, δ, ϵ.

- The topological billiard α consists of two billiards glued along a common arc of an ellipse. One of the billiards is bounded by an ellipse arc (with the parameter $\lambda = 0$), two arcs of hyperbolas, and a focal line. The other billiard is bounded by two arcs of the same hyperbola and two arcs of ellipses (with the parameters $\lambda = 0$ and $\lambda = \frac{b}{2}$).
- The topological billiard β is a disjoint union of two billiards α.
- The topological billiard γ is obtained by gluing two copies of the billiard α along the focal line.
- The topological billiard δ is obtained by gluing two copies of the billiard α, simultaneously along the arc of the smaller ellipse with the parameter $\lambda = \frac{b}{2}$, and the focal line.

- The topological billiard ϵ consists of two copies of one billiard glued along the common arc of the ellipse and the segment of the focal line. Every copy is bounded by an arc of an ellipse (with the parameter $\lambda = 0$), two arcs of hyperbolas, and the focal line.

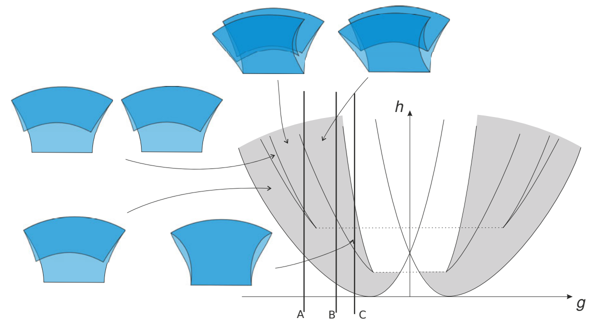

Fig. 7. One of the cases of the bifurcation diagram of the Zhukovsky case. In dark color, the chambers of the diagram whose Liouville foliation of isoenergy surfaces can be modelled by integrable billiards are distinguished. These chambers are marked by Greek letters. The corresponding integrable billiards are marked by the same letters. Vertical lines A, B, C encoding the symplectic sheets marked by the same symbols are also indicated.

6.1.1. Construction of the force billiards $J_A,\ J_B,\ J_C$. Let us describe the force billiards $J_A,\ J_B,\ J_C$ partially modelling the integrable system of Zhukovsky's case on four-dimensional symplectic leaves $A,\ B,\ C$.

- The symplectic leaf A corresponds to a straight line passing sequentially through the chambers $\alpha,\ \beta,\ \gamma,\ \delta$ of the bifurcation diagram. The initial state of the force billiard J_A is the billiard α, and the final state is the billiard δ. At the first energy jump, the billiard α joins to its copy, which is glued to the original billiard at the next two jumps. At the beginning, the gluing takes place along the focal line, and then along the arc of the smaller ellipse with the parameter $\lambda = \frac{b}{2}$.

- The symplectic leaf B passes sequentially through the chambers $\alpha,\ \gamma,\ \delta$ of the bifurcation diagram. The initial state of the force billiard J_A is the billiard α, the intermediate state is the billiard γ, and the final state is the billiard δ. Unlike the previous force billiard, at the first jump, there is a simultaneous occurrence of a copy of the billiard α and the gluing of two copies along the focal line.

- The symplectic leaf C corresponds to a straight line passing sequentially through the chambers $\alpha,\ \epsilon,\ \delta$ of the bifurcation diagram. When moving in the chamber α, the billiard of the same name is changed in such a way that the parameter λ, which is initially equal to $\frac{b}{2}$, tends to the value b. In the limit, when reaching the chamber wall of the diagram, the billiard state is two copies of the "bottom" sheet of the billiard α. At the moment of the jump, these two copies are glued together along the focal line. At the moment of the next jump, two billiard sheets are added, glued along the arc of the ellipse with parameter $\frac{b}{2}$, which are glued to the previous billiard state along the arc of the larger ellipse. Here one convex gluing is replaced by two convex ones and one nonconvex gluing. On the other hand, this jump can be interpreted as an "extruding of a fold" from two new small billiards inside the billiard (see the form of the billiards ϵ and δ in Fig. 7), in which the billiard in the entire small vicinity of the jump remains homeomorphic to an annulus, and only the number of its gluing increases.

6.2. Zhukovsky system: the case of the bifurcation diagram c

Let us now consider case *c)* of the bifurcation diagram of the Zhukovsky system. Consider new chambers $\theta,\ \eta$, and ζ.

To the chamber θ, there corresponds a Liouville foliation, which, in the class of integrable topological billiards, can be modelled only by billiards of the following form. Consider a triangular domain on a plane bounded by an ellipse arc, a convex hyperbolic arc, and a focal line. Glue two copies of such a domain along the arc of the hyperbola and along the arc of the focal line. The resulting billiard implements the required foliation. The gluing (as in Fig. 8) of a "collar" along the arcs of ellipses does not change the Liouville foliation (see [12, 13]). A similar foliation is obtained by replacing the glueings along arcs of hyperbolas by glueings along arcs of ellipses and vice versa. Note that in other billiards, including the class of billiard books, and in billiards bounded by arcs of circles, no similar foliation occurred.

To the chamber η, there corresponds the Liouville foliation encoded by the molecule $A - A$ with the mark $r = \frac{1}{2}$. This foliation occurs in billiards in two cases. First, this happens for the case in which a topological billiard contains convex glueings only, is homeomorphic to a disk, and contains only one conical point not lying on the focal line. It is an obstacle to include this billiard into a force billiard containing a billiard modelling the foliation in the chamber θ. The point is that the billiard in the chamber θ contains a gluing along the focal line along which a cut will needed. However, in the standard definition of force billiards, this operation was prohibited. The other case of occurrence of such a foliation happens in a topological billiard obtained by gluing two circular billiards. Obviously, this billiard cannot either be a part of a force billiard containing a billiard modelling the foliation in the chamber θ. Therefore, in Fig. 8, we highlighted this chamber in a darker color.

To the chamber ζ there corresponds the Liouville foliation encoded by the molecule $A-A$ with the marks $r=\infty$, $\varepsilon = 1$. This foliation occurs in the class of topological billiards or billiard books only for a billiard in an annulus bounded by two concentric circles and does not occur in billiards bounded by arcs of ellipses and hyperbolas. In Fig. 8, we highlighted this chamber in a darker color, since the depicted billiards are bounded by arcs of ellipses and hyperbolas. Note that, if we want to include the billiard simulating the foliation in the chamber θ in the force billiard we are constructing, then we are to stay within ellipses and hyperbolas as billiard boundaries. If we want to include into a force billiard both the billiard modelling the foliation in the chamber ζ and the billiard modelling the foliation in the chamber θ, then we are to pass somehow from boundaries lying on circles to boundaries lying on nontrivial ellipses and hyperbolas.

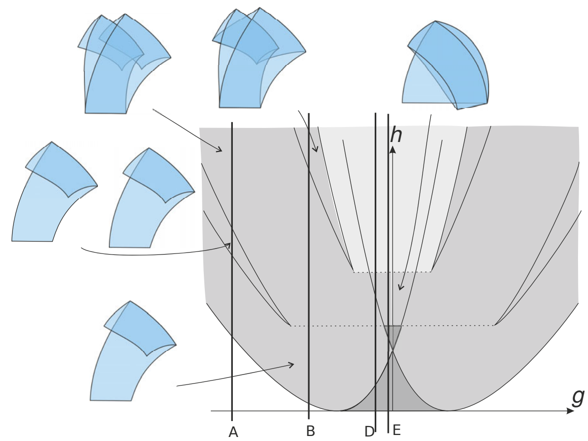

Fig. 8. One of the cases of the bifurcation diagram of the Zhukovsky case. The chambers of the diagram whose Liouville foliation of isoenergy surfaces can be modelled by integrable billiards are highlighted in darker colors. These chambers are markled with Greek letters and the same letters mark integrable billiards. The vertical lines A, B, D, E encoding symplectic sheets of the same name are also indicated.

6.2.1. Construction of the force billiards J_D and J'_D and the billiards J^i_E, $i \in \{1..4\}$. Let us now describe the force billiard J_D partially modelling the symplectic sheet D of Zhukovsky's case. The simulation is carried out in three chambers α, γ, and θ. The billiard states are described above. The first jump occurs in the same way as in the billiard J_B : there is a simultaneous occurrence of a copy of the billiard α and gluing two copies along the focal line. On the next jump, the following events occur. First, the arc of a nonconvex hyperbola lies on the focal line and the gluing takes place along this line. Second, a pairwise gluing of equivalent billiards occurs along the arc of the convex hyperbola. The result is the billiard θ modelling the foliation in the chamber of the same name.

Let us describe the force billiard J'_D, which also partially simulates the symplectic sheet D of Zhukovsky's case on more regular surfaces than the billiard J_D does. To obtain this billiard, we, first, partially admit the ungluing of billiards at a jump (but keeping the condition that the sheets remain subsets of sheets of greater energy). Secondly, let us permit the quadric segments to change their parameters. More exactly, at the moment of one of the jumps, we permit the replacement of the boundary circles and straight lines of the billiards by a family of confocal ellipses and hyperbolas. Obviously, the inverse transformation equalizes the parameters a and b of the family, which are the roots of the semiaxes of the family of confocal quadrics. The billiard states are shown in Fig. 9.

The first three states of this billiard are bounded by arcs of concentric circles and segments of straight lines passing through the common center of these circles.

- The initial state of the billiard is two billiards bounded by two concentric circles (in what follows, we assume that the common center of these circles is the origin) and two coordinate lines Ox and Oy, see Fig. 9. These billiards are glued along common arcs of straight lines into an annulus. This ensures the mark $r=\infty$ in the

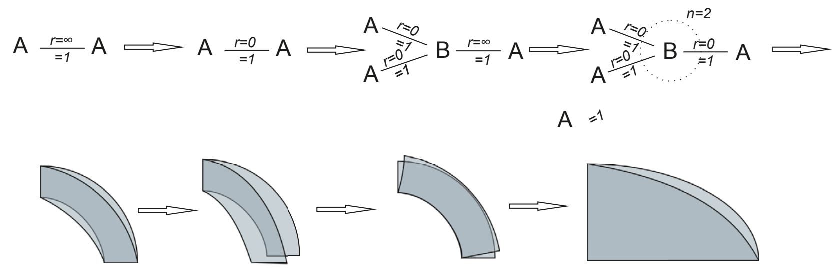

Fig. 9. Another way to partially simulate an integrable system of the Zhukovsky case by force billiards on a 4-dimensional symplectic submanifold corresponding to the sheet D. The force billiard J'_D described above is shown.

corresponding Liouville foliation between the atoms A. There are no other atoms in the coarse molecule, since there are no nonconvex glueings, and this leads to the fact that all regular fibers are homeomorphic to tori. Two singular fibers (motions along the larger circle) are directed oppositely. This leads to the fact that the mark ε between the atoms A atoms is 1.

- The second billiard state is obtained from the previous one by ungluing along the segment of the straight line Ox, see Fig. 9. The billiard becomes homeomorphic to the disk. This leads to the fact that the corresponding Liouville foliation is described by the molecule AA with the mark $r = 0$.
- The third state is obtained from the second by the ungluing along the segment of the straight line Oy and the nonconvex gluing along an arc of a smaller circle, see Fig. 9. The presence of a nonconvex gluing leads to the formation of an atom B in the corresponding Liouville foliation. The marks are calculated similarly to the case of the billiard bounded by arcs of confocal quadrics. In what follows, we assume that, when moving along the straight line D in this chamber, the radius of the smaller circle is reduced to zero, while remaining nonzero inside the chamber.
- The last billiard state is obtained from the third one by the following transformation. The radius of the smaller circle becomes equal to zero; after this, a gluing along the coordinate lines Ox and Oy takes place. Then there is a stretching along the axis Ox corresponding to the replacement of the boundary circle by an ellipse. The straight line Ox becomes the focal line, see Fig. 9.

Similarly to the construction concerning the sheet D, we construct a force billiard partially modelling the symplectic leaf E. We are interested in the evolution of the Liouville foliations that corresponds to the sequence of chambers $\zeta,\ \alpha,\ \eta,\ \theta$. The four ways are shown in Fig. 10. Let us explain each of them.

1. The first way is simulation in the chambers $\zeta,\ \alpha,\ \eta$. This sequence of foliations arises in the Lagrange case and was shown in the paper [32]. The initial state of the billiard is a topological billiard obtained by gluing two copies of flat billiard-annuli bounded by concentric circles. When moving in the first chamber, the radius of the smaller circle of one of the two flat billiards decreases and contracts to a point at the first jump. When moving in the next chamber, the remaining boundary of the billiard contracts to a point. As a result, in the third chamber, we obtain a topological billiard glued from two disks. In Fig. 10, these billiard states are schematically shown as domains on a two-dimensional sphere. The resulting force billiard is denoted by J^1_E.
2. The second way is modelling in the chambers $\alpha,\ \eta,\ \theta$ in such a way that the circles are replaced by ellipses in the moment of the last jump. However, under this evolution, no ungluing occurs at any stage. The initial billiard state is the two quarters of the disk (which is bounded by a circle) glued along an arbitrary line segment of the boundary (without loss of generality, this is a segment of the Oy axis). At the first jump, these two quarters are glued along another line segment of the boundary. These billiards are homeomorphic to disks, contain no nonconvex gluings, and the second billiard contains one conical point. This enables us to claim that their Liouville foliations are described by the indicated Fomenko–Zieschang invariants. At the moment of the second jump, there is a transformation of the family of concentric circles and lines through the origin into a family of confocal quadrics. As a result, the billiard thus obtained has the desired foliation. The resulting force billiard is denoted by J^2_E.
3. The third way also enables one to partially simulate the symplectic leaf E in chambers $\alpha,\ \eta,\ \theta$, without going beyond the confocal family of ellipses and hyperbolas. However, in this case, it is necessary to use the partial ungluing of the billiard spines. Consider a flat billiard B_0 disjoint from the focal line and bounded by two arcs of

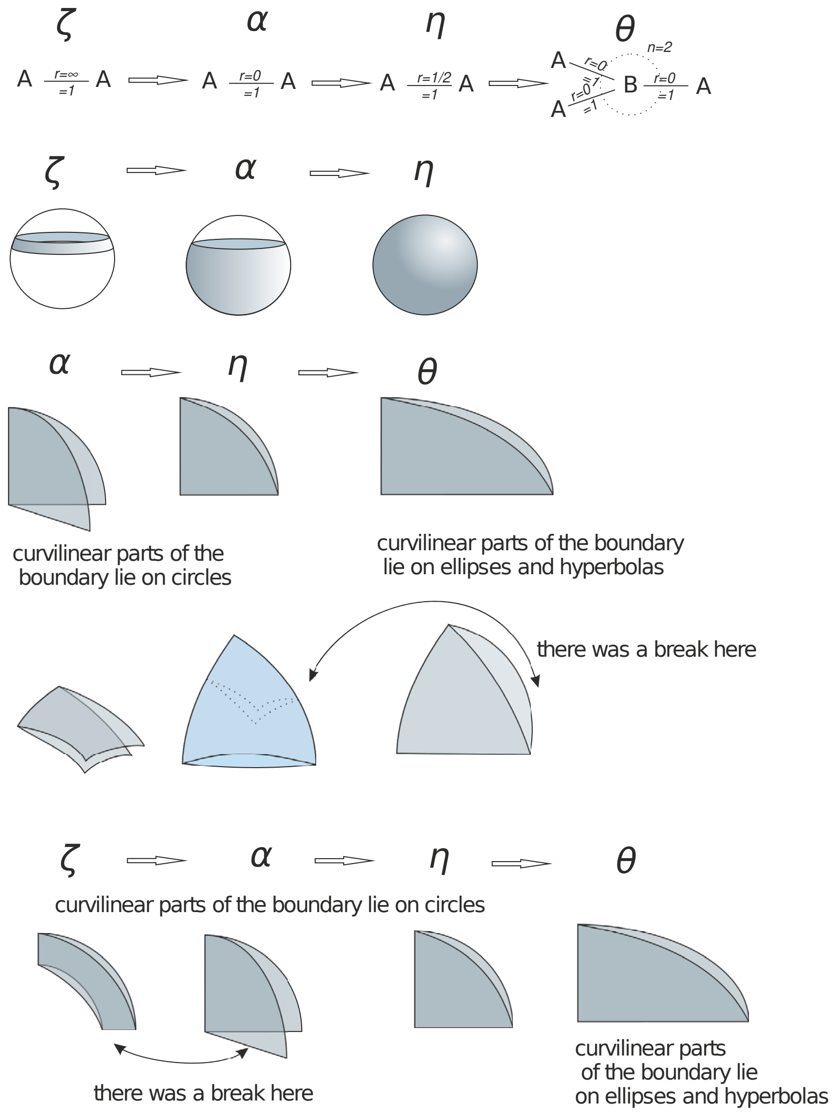

Fig. 10. Diverse ways to partially simulate the integrable system of the Zhukovsky case by force billiards on a 4-dimensional symplectic submanifold corresponding to the sheet E.

hyperbolas (convex and nonconvex) and two arcs of ellipses. The initial state of the force billiard is the topological billiard obtained by gluing two copies of the flat billiard B_0 along the arc of the convex hyperbola. When moving in the chamber α, the parameter of the boundary hyperbola decreases and the parameter of the smaller boundary ellipse increases in such a way that, in the moment of the jump, the corresponding arcs lie on the focal line. At the moment of this jump, the gluing along the boundary arcs lying on the arc of the ellipse also occurs. The billiard thus obtained contains a conical point. On the next jump, the gluing along the focal line and the ungluing along the arc of the ellipse take place. The force billiard thus obtained is denoted by J_E^3.

4. The fourth method enables us to cover all four chambers ζ, α, η, θ simultaneously (see the bottom line of Fig. 10). The initial billiard state is two billiards bounded by two concentric circles and two coordinate lines Ox and Oy. These billiards are glued along the common arcs of straight lines into an annulus. When moving in the first chamber ζ, the radius of the smaller boundary circle reduces to zero. At the moment of the first jump, the two quarters (thus obtained) of the disk must be unglued along the segment of the straight line Ox and, at the moment of the second jump, these quarters must be glued back along the same boundary. At the moment of the third jump, the already described "pulling" procedure takes place, in which the arc of a circle is replaced by an arc of an ellipse. The force billiard thus obtained is denoted by J_E^4.

Theorem 2. *The constructed integrable evolutionary billiards J_A, J_B, J_C, J_D, as well as the billiards J_D', J_E^i, $i \in \{1 \dots 4\}$ implement (in the sense of the Liouville equivalence) the integrable Zhukovsky case on the part of the phase* *symplectic manifold M_g^4 corresponding to the lines of the same name in Figs. 7, 8.*

Remark 9. We stress that the evolution of billiard walls occurs in the class of confocal quadrics, which ensures the integrability of the system at every moment of its evolution, at all successively occurring isoenergy 3-surfaces with increasing energy.

Proof. We rely on the results of nontrivial calculations made in the works cited below. The proof of the theorem follows from the comparison of the known Fomenko–Zieschang invariants for the Zhukovsky case, which were calculated by A. A. Oshemkov, and the Fomenko–Zieschang invariants of the billiard states of the force billiards described above. We omit the specific list of invariants and their comparisons in view of large size.

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