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Main Directions and Achievements of the Chair of Differential Geometry and Applications at the Present Stage

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Abstract—Recent results by the staff of the Chair of Differential Geometry and Applications in various areas of geometry and its applications are briefly presented.

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In this survey we highlight key studies of the members (as of 2024) of the Chair of Differential Geometry and Applications. The website of the Chair presents a detailed historical survey of achievements of the Chair[1], a list of monographs written by the staff[2], slides of the talk by A.T. Fomenko and V.A. Kibkalo devoted to the 100-year anniversary of V.F. Kagan scientific school[3]. The history of the Chair was also outlined in the monograph *80 Years of the Faculty of Mechanics and Mathematics of the Lomonosov Moscow State University* (Moskovskii Gosudarstvennyi Universitet, Moscow, 2013, pp. 84–91) on the occasion of the 80-year anniversary of the faculty.

1. PLATEAU PROBLEM

A.T. Fomenko solved the well-known multidimensional Plateau problem in the class of spectral manifolds. In particular, let a "contour" A be given, that is, a $(k-1)$-dimensional closed manifold in an ambient Euclidean space (or, more generally, a submanifold bordant to zero in the Riemannian space is given). Then there always exists a k-dimensional stratified globally minimal "surface" W with a boundary A parameterized by the spectrum (sequence) of k-dimensional manifolds W_i with the boundary A. In addition, any nontrivial element of the group of spectral bordisms of the Riemannian manifold is realized by a globally minimal "spectral surface"—a closed cycle (see [1]). A similar solution to the multidimensional Plateau problem was obtained both for bordisms and for an arbitrary extraordinary theory of spectral (co)homologies, for instance, for a K-functor (for a K-theory of vector foliations). A development of this topic at the chair has been presented in the works by A.O. Ivanov and A.A. Tuzhilin (see Section 5).

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[1] Research achievements of the members of the chair. http://dfgm.math.msu.su/sci_archiev.php.
[2] Monographs of the members of the chair. http://dfgm.math.msu.su/books.php.
[3] A.T. Fomenko and V.A. Kibkalo, Presentation of the Chair of Differential Geometry and Applications. http://dfgm.math.msu.su/files/Dfgm90-100-2023.pdf.

2. ARGUMENT SHIFT METHOD. MISHCHENKO–FOMENKO CONJECTURE

In 1978, A.S. Mishchenko and A.T. Fomenko introduced a general argument shift method on Lie algebras (see [2]). For any Lie algebra $\mathfrak{g}$, element $\xi \in \mathfrak{g}^*$, and two polynomials f and g on $\mathfrak{g}^*$ that are invariant with respect to conjugation, the iterated derivatives of f and g in the direction ξ commute with respect to the Poisson bracket: $\{\xi^p(f), \xi^q(g)\} = 0$ for all $p, q \geqslant 0$. A Poisson-commutative subalgebra A_ξ generated by elements $\xi^p(f)$ for such f and $p \geqslant 0$ in the algebra of polynomial functions on $\mathfrak{g}^*$ is called the *argument shift algebra*, or the *Mishchenko–Fomenko algebra*.

A.S. Mishchenko and A.T. Fomenko discovered multiparametric polynomial families of completely integrable in the Liouville sense Hamiltonian systems (IHSs) on all reductive Lie algebras, in particular, on all semisimple ones. For an arbitrary finite-dimensional Lie algebra, the conjecture about the existence of such polynomial IHSs is known as the *Mishchenko–Fomenko conjecture*. Subsequent works of many mathematicians strengthened confidence in its truth. Finally, the conjecture in full was algebraically proved by S.T. Sadetov for Lie algebras over the field of characteristic 0. This is an extremely nontrivial theorem. After that, the proof in terms of Poisson geometry was suggested by A.V. Bolsinov.

These results are intensely developed today by many mathematicians. For instance, for semisimple $\mathfrak{g}$ the algebra A_ξ "rises" (L.G. Rybnikov, 2006) to a universal enveloping algebra as its commutative subalgebra: $\hat{A}_\xi \subseteq U\mathfrak{g}$. However, approaches for constructing the algebra $\hat{A}_\xi$ known before do not enable one to find the operator $\hat{\xi} : [\hat{\xi}^p(f), \hat{\xi}^q(g)] = 0, \forall p, q \geqslant 0$ and $\forall f, g \in U\mathfrak{g}^G$. For $\mathfrak{g} = \mathfrak{gl}_n$ it became possible to construct this operator as a linear combination of "quasi-derivatives" $U\mathfrak{gl}_n$ (introduced by D.I. Gurevich, P.N. Pyatov, and P.A. Saponov in 2012).

Theorem (Ikeda and Sharygin) [3, 4]. *For a numerical matrix $\xi = \left(\xi_i^j\right)$, from $\mathfrak{gl}_n^*$ and an operator $\hat{\xi} = \sum_{i,j=1}^n \xi_i^j \hat{\partial}_j^i$, where $\hat{\partial}_j^i$ are partial quasi-derivatives, it is true that $[\hat{\xi}^p(f), \hat{\xi}^q(g)] = 0$ for all $f, g \in U\mathfrak{g}^G$ and $p, q \geqslant 0$.*

3. COMPLETELY INTEGRABLE IN THE LIOUVILLE SENSE HAMILTONIAN SYSTEMS

Completely integrable in the Liouville sense Hamiltonian systems are given by a Hamiltonian field $v = \operatorname{sgrad} H$ on a symplectic manifold (M^{2n}, ω) and have n first integrals $f_1, \dots, f_n$, involutive $\{f_i, f_j\} = 0$, and functionally independent. Our colleagues study the topology of their Liouville foliations (M^{2n} is a disjoint union of connected common level surfaces of first integrals $f_1, \dots, f_n$), singularities arising in them, the dynamics of phase trajectories lying on fibers, and algebraic structures associated with the integrability.

3.1. Noncommutative Integrability

A.S. Mishchenko and A.T. Fomenko discovered [5] a broad class of noncommutative integrable Hamiltonian systems (the Lie algebra of their integrals is noncommutative, that is, $\exists i \neq j : \{f_i, f_j\} \neq 0$) and demonstrated how to reduce them to the involutive case. These are geodesic flows of many left-invariant metrics on Lie groups and symmetric spaces.

Theorem (Mischenko–Fomenko) [5]. *Let a Hamiltonian system completely integrable in the noncommutative sense be given on a symplectic manifold. Then it is completely integrable in the Liouville sense and in the common sense, and the commuting integrals of the new algebra of integrals can be functionally (and algorithmically) expressed via previous noncommuting integrals.*

3.2. Topological Classification of Integrable Hamiltonian Systems

A.T. Fomenko constructed a theory of topological classification of integrable Hamiltonian nondegenerate (in the Bott sense) systems with two degrees of freedom. Two systems are called Liouville equivalent if their Liouville foliations are fiberwise homeomorphic, that is, the closures of their generic solutions are identified by a homeomorphism. The Fomenko—Zieschang invariant classifies [6] systems in nonsingular energy zones (on nonsingular isoenergy surfaces Q^3). The edges of this graph correspond to the one-parameter families of regular tori, the vertices correspond to the singularities (Fomenko 3-atoms), and the numerical marks encode gluing of Q^3 from the set of 3-atoms along their boundary tori.

The topological invariants have been computed by A.T. Fomenko and other mathematicians for many concrete IHSs from the geometry, mechanics, and physics [7]. A.A. Oshemkov has studied integrable cases in the rigid-body dynamics, the Sokolov case on the Lie algebra $so(4)$, the problem of two centers on a sphere, the Adler—van Moerbeke case, and V.A. Kibkalo has studied multiparametric analogs of the Kovalevskaya top on Lie algebras $so(4)$ and $so(3, 1)$, as well as the pseudo-Euclidean analogs of mechanical systems. Many of these systems are bi-Hamiltonian, that is, the integrals commute with respect to two compatible Poisson brackets. Several general results on the singularities of such systems were obtained by A.V. Bolsinov and A.A. Oshemkov. E.A. Kudryavtseva, together with D.A. Fedoseev and O.A. Zagryadskii, described all superintegrable mechanical systems that are rotation-invariant. All these wide results are the basis for the fundamental topological atlas of IHSs created by the Chair staff [7]. Interesting and unexpected topological relations between systems of different origin were revealed.

A.T. Fomenko, S.V. Matveev, H. Zieschang, and A.V. Brailov proved coincidence of the class of nonsingular energy levels of IHSs and Waldhausen graph-manifolds (Seifert manifolds are their subclass). In the theory of hyperbolic manifolds, the Fomenko—Matveev—Weeks 3-manifold is known that has the smallest volume among all hyperbolic closed orientable 3-manifolds [8].

3.3. Nondegenerate Singularities of Integrable Hamiltonian Systems and Morse Functions

Almost all singularities of the studied IHSs turned out to be nondegenerate (in the Bott sense). For a typical singularity of IHS on M^{2n}, the restriction of an appropriate integral to the transversal to the critical submanifold is a Morse function. A.T. Fomenko classified Bott 3−*atoms* of integrable systems $v = \mathrm{sgrad}\, H$ on M^4, that is, Liouville foliations near the fiber with critical nondegenerate circles S^1_i. The atom has a structure of Seifert S^1-foliation with a two-dimensional base and several its fibers can be special of the type (2,1) (see monograph [7]).

The critical points of the Morse function H on M^2 are either center (maximums and minimums of H) or saddle points. A special fiber with saddle points is a finite graph with vertices of degree 4 embedded in M^2. The foliation in an invariant neighborhood of a special fiber is called 2-atom. A.A. Oshemkov constructed (1994) a combinatorial invariant—an f-graph, useful in description of 3-atoms, classification of simple Morse functions, and construction of invariants of Morse—Smale systems on surfaces M^2 and their further generalizations (1998). Note also the relation between the 2-atoms and virtual knots in thickened surfaces $M^2 \times I$.

Nondegenerate singularities of rank $r < n - 1$ of IHSs on M^{2n} were studied by many mathematicians. A.A. Oshemkov developed [9] semilocal classification of saddle singularities of rank 0 in terms of f_n-graph introduced by him. This topological invariant is a generalization of f-graph of saddle 2-atoms V_i. In this work, A.A. Oshemkov also constructed an algorithm for computation of components (2-atoms $V_1, \dots, V_n$) of the minimal model of a singularity by its f_n-graph and estimating the complexity of 2-atoms V_i. The efficiency of the approach was demonstrated by creating a list of singularities of small complexity [10]. A semilocal classification of saddle—focus singularities of IHSs on M^6 was constructed by A.A. Oshemkov and I.K. Kozlov [11].

E.A. Kudryavtseva and A.A. Oshemkov (2022) obtained a combinatorial sufficient condition for structural stability of nondegenerate singular fibers of Lagrange foliations. A.A. Oshemkov (2018) obtained a stability criterion of saddle singularities of rank 0 under integrable perturbations.

A.A. Oshemkov investigated the complex of singularities of IHSs on M^{2n}. The cycles given by singularities of rank r of IHSs turned out to be Poincaré dual to the corresponding Chern classes TM^{2n}. By homological properties of the complex, he composed a list of nondegenerate IHSs possible on "simple" M^{2n}.

3.4. Degenerate Singularities

E.A. Kudryavtseva obtained a symplectic (together with A.V. Bolsinov and L. Guglielmi [12]) and a C^∞-smooth (together with N.N. Martynchuk) classifications of parabolic orbits of Lagrangian foliations. Such singularities were found in mechanical systems: the Kovalevskaya top (V.A. Kibkalo, E.A. Kudryavtseva) and the axisymmetric Zhukovsky system (V.A. Kibkalo).

E.A. Kudryavtseva studied hidden Hamiltonian toric symmetries of singular orbits of IHSs, obtained the sufficient conditions of existence and rigidity of such symmetries, and classified Hamiltonian actions of torus near its compact singular orbits.

As an application, C^∞-smooth classifications of singular orbits of Lagrangian foliations on (M^6, ω) were obtained (together with M.V. Onufrienko and L.M. Lerman for orbits of corank 1 and 2, respectively). Furthermore, they discovered and classified new types of singularities of corank 2: with Williamson saddle—saddle (with the hyperbolic $m : n$ resonance and, perhaps, twisting) and focus—focus types. The structural stability was proved for all the types at small real-analytical integrable perturbations.

3.5. Integrable Billiards, Quadrics, and Topological Modeling of Integrable Hamiltonian Systems

An important class of IHSs is geodesic flows on quadrics and billiards in domains bounded by them. Trajectory of a billiard in such domain has a caustic (a line tangent to each link of the trajectory). The caustic is confocal to the boundary, which follows from the prominent Jacobi—Chasles theorem on integrability of a geodesic flow on an ellipsoid. Recently, the class of the latter was substantially extended: V.V. Vedyushkina introduced [13] billiard books (CW-complexes glued from flat tables and equipped with commuting permutations), A.T. Fomenko introduced billiards with slipping (2021) and force evolutionary billiards (2022). They can also be equipped with a magnetic or potential field. Here, the dynamics of a ball on a "complex" table can visually and efficiently model [14] very complicated multidimensional IHSs from physics, mechanics, and geometry. It describes the evolution of systems and their singularities.

Computation of Fomenko—Zieschang invariants of flat integrable billiards began in 2009 by V. Dragović and M. Radnović. In 2015, V.V. Vedyushkina (Fokicheva) introduced topological billiards (a subclass of the billiard book class, which are homeomorphic to 2-manifolds) and performed their structural and topological classification, in the case of both confocal ellipses and hyperbolas and confocal parabolas.

Conjecture (Fomenko, 2019). *A suitable class of integrable billiards can topologically model nondegenerate IHSs on M^4 in nonsingular energy zones (on nonsingular Q^3_h: $H = h$). In particular, the class of "billiard" Liouville foliations includes all Bott 3-atoms, all bases of Liouville foliations on Q^3 with such 3-atoms, all homeomorphism classes of isoenergy manifolds Q^3 possible in IHSs, and all classes of Liouville equivalence of nondegenerate IHSs on Q^3 classified by Fomenko—Zieschang invariants.*

Several statements of the Fomenko billiard conjecture have been already proved. For all Bott 3-atoms [13] and bases of Liouville foliations (2021), V.V. Vedyushkina and I.S. Kharcheva algorithmically constructed billiard books with such foliations on Q^3. For each connected sum of lense spaces $L(q_i, p_i)$ and direct products $S^1 \times S^2$, V.V. Vedyushkina constructed books with such homeomorphism class of Q^3 (2021), that is, the class Q^3 of billiards appeared to be not limited by Seifert manifolds: the latter does not include the manifold $\mathbb{R}P^3\#\mathbb{R}P^3\#\mathbb{R}P^3$, where $\mathbb{R}P^3 \cong L(2, 1)$. V.V. Vedyushkina and V.A. Kibkalo proved several statements of the "local" version of the Fomenko conjecture on realization by billiards of marked "subgraphs" of an invariant, including all values of numerical marks r, ε, n (see [15]). Realizability of marks and 3-atoms means realizability of all "elements" of the Fomenko—Zieschang invariant by billiards.

Foliations of billiard books turned out to be Liouville equivalent to many classical integrable systems of geometry and mechanics. V.V. Vedyushkina and A.T. Fomenko (2019) modeled all geodesic flows on orientable surfaces with integrals of degree 1 and 2. The Euler and Lagrange tops are completely realized by a set of topological billiards. In all energy zones at once, such a top is realized by a force evolutionary billiard, which results in "billiard equivalence" of these two classical tops (2022). The Zhukovsky, Kovalevskaya, and Goryachev–Chaplygin systems are modeled by billiards in some energy zones, which provides examples of decreasing the degree (in momenta) of the integral from 4 and 3 to 2.

Billiards on multidimensional confocal tables are intensely studied by G.V. Belozerov [16]. He studied the topology of foliations and their singularities for many such domains in $\mathbb{R}^3$ and determined the homeomorphism class of their isoenergy 5-surfaces: the sphere S^5, the direct products of spheres $S^1 \times S^4$ and $S^2 \times S^3$. V.A. Kibkalo proposed a multidimensional generalization of confocal billiard books and checked the correctness of gluing in the neighborhood of cells of codimension 2 and higher codimensions.

The Jacobi–Chasles theorem states that in the Euclidean space $\mathbb{R}^n$ all tangent straight lines drawn to geodesic on an n-axial ellipsoid are simultaneously tangent to $(n-2)$ quadrics confocal with the ellipsoid. V.A. Kibkalo hypothesized the integrability of a geodesic flow on intersection of several confocal quadrics and proved it in the case of two-dimensional intersection of $(n-2)$ quadrics in $\mathbb{R}^n$. G.V. Belozerov proved a complete analog of the Jacobi–Chasles theorem for any intersection of k nondegenerate quadrics in $\mathbb{R}^n$ (see [17]), and he also obtained a condition for the potential field sufficient for integrability of the system at such intersection.

The recent results of G.V. Belozerov and A.T. Fomenko prove the generalized Jacobi–Chasles theorem in Minkowski spaces and spaces of constant sectional curvature. A remarkable consequence of this theorem is the integrability of confocal billiards on any intersection of confocal quadrics. It was also shown that in the two-dimensional case the latter cannot be extended to surfaces locally non-isometric to spaces of constant sectional curvature.

3.6. Integrable Hamiltonian Systems with Noncompact Fibers

Integrable Hamiltonian systems with noncompact Liouville foliations and incomplete flows of Hamiltonian vector fields were studied in the works of our colleagues and descendants (D.V. Novikov, K.R. Aleshkin, D.A. Fedoseev, and T.A. Lepskii). A comprehensive review was carried out by A.T. Fomenko and D.A. Fedoseev (2016). The noncompact foliations of billiards on unbounded tables were described by V.V. Vedyushkina and A.T. Fomenko (2017).

E.A. Kudryavtseva proved an analog of the Liouville theorem for a subclass of IHSs on M^4 with incomplete flows, including systems on $(\mathbb{C}^2, \mathrm{Re}\,(dz \wedge dw))$ with integrals $(\mathrm{Re}\, f(z,w),\ \mathrm{Im}\, f(z,w))$ for a holomorphic function f.

V.A. Kibkalo studies pseudo-Euclidean analogs of mechanical systems introduced by A.V. Borisov and I.S. Mamaev (Russ. J. Math. Phys., 2016). For such analogs of the Kovalevskaya top, he proved [18] a compactness criterion for a common level surface of the energy, first integral, and Casimir functions in $\mathbb{R}^6$. For such analogs of the Euler system [19] and the axisymmetric Zhukovsky system (2024), he constructed bifurcation diagrams, described bifurcations (analogs of Fomenko 3-atoms, including noncritical ones) and bases of Liouville foliation on nonsingular Q^3.

3.7. Orbital Invariants of Integrable Hamiltonian Systems

A.V. Bolsinov and A.T. Fomenko discovered invariants of a continuous orbital equivalence of IHSs and computed them, for instance, for the Euler top from the rigid-body dynamics in $\mathbb{R}^3$ and for the Jacobi system, i.e. for geodesic flow on an ellipsoid. These well-known systems turned out to be not only Liouville, but also continuously orbital equivalent, although there is no smooth orbital equivalence and topological conjugacy [7]. Recently, A.T. Fomenko and G.V. Belozerov computed edge orbital invariants of many integrable billiards (2024).

E.A. Kudryavtseva constructed infinite series of bicyclic and tricyclic saddle 2-atoms (3-atoms), on which there exist invariants of topological conjugacy (orbital invariants) stable under "Hamiltonian" (integrable) perturbations of splitting of a complicated atom into a molecule with simple atoms. She showed the absence of such invariants for plane atoms.

3.8. Nijenhuis Geometry and Integrable Systems

Manifolds equipped with an Nijenhuis operator field with zero torsion arise in various problems of algebra and geometry. Their systematic study began in the end of 2010s by A.Yu. Konyaev together with A.V. Bolsinov and V.S. Matveev, which turned out to be extremely fruitful [20]. They succeeded to construct a local theory of Nijenhuis operators. In the previously known cases the algebraic type of operator does not change and/or was hyperbolic. It appeared that for a wider class of operators (gl-regular Nijenhuis operators) there exists a theory of normal forms. Their complete list was obtained in [21] for the dimension 2 case.

The relation of the Nijenhuis operators and their singular points (points of scalar type) with left-symmetric algebras was established and the problem of linearization for these singularities was formulated. In the two-dimensional case, A.Yu. Konyaev obtained a complete classification of nondegenerate left-symmetric algebras [22].

Integrable systems (ISs) turn out to be the main applications of the Nijenhuis geometry. The operators arise in systems integrable in quadratures. Applying the Nijenhuis operators enable to reformulate in geometric terms the separation of variables, at which the class of integral Hamilton–Jacobi equations extends significantly. The classical results of A.V. Bolsinov, V.S. Matveev, and Pucaccio imply that all two-dimensional ISs in the neighborhood of a generic point are obtained by exactly this method.

The relationship between the Nijenhuis operators and quasi-linear ISs was established. Many classes of such systems were constructed using Nijenhuis operators. In particular, such approach allowed obtaining non-diagonalizable analogs of weak nonlinear Ferapontov systems and integrating them in quadratures.

A special class of weak nonlinear systems associated with the Killing tensors of Benenti systems turned out to be related to the problem of constructing finite-zone potentials of the Hill equation. Moreover, their solutions are exactly the finite-zone solutions to the KdV, Camassa–Holm, and other equations.

4. GEOMETRY AND MATHEMATICAL PHYSICS

Research in the field of mathematical physics at our Chair is conducted by A.I. Shafarevich and his descendants. A.I. Shafarevich is a well-known expert in the field of asymptotic and geometric theory of linear and nonlinear partial differential equations, quantum mechanics, and fluid dynamics. He solved the problem of multiphase asymptotics for the fluid dynamics equations formulaed by V.P. Maslov and multiply discussed in the scientific literature. Furthermore, he profoundly developed the methods for studying asymptotic solutions of nonlinear partial differential equations; for the fluid dynamics equations he established the relation between asymptotics and topological invariants of vector fields and Liouville foliations [23]; he also obtained interesting applications to fluid dynamics (including the asymptotic description of the evolution of localized structures in incompressible fluid).

In addition to that, A.I. Shafarevich investigated the theory of quasi-classical quantization of invariant isotropic manifolds of Hamiltonian systems. He described spectral series that correspond to trajectories and invariant tori in resonant systems, as well as to one-dimensional singular invariant sets in partially integrable systems. A.I. Shafarevich obtained several results on quantization of complex manifolds, described spectral series of Calogero–Strocchi quantum systems and non-self-adjoint periodic Schrödinger operators with a complex potential [24].

A.I. Shafarevich also studied the spectral theory and the theory of evolution equations on some classes of CW complexes: geometric graphs and hybrid spaces—decorated graphs obtained from graphs by replacement of vertices on manifolds [25]. Recently, he have obtained several results about the quasi-classical spectrum of Schrödinger operators on such spaces and about the time evolution of solutions to nonstationary problems. In particular, he described the statistics of propagation of localized solutions to the nonstationary Schrödinger equation on a geometric graph at large times; he established the relation between such statistics and the well-known problems and results of the analytic number theory.

Another direction of research of A.I. Shafarevich is the generalization of the quasi-classical Maslov theory to the case when the coefficients of the equations and the manifolds on which these equations are given include singularities. He characterized spectral series of Laplace and Schrödinger operators with delta potentials and on the surfaces with conic points [26]; he described modifications of the Lagrange manifolds corresponding to quasi-classical solutions to Schrödinger equations and short-wave solutions to general strictly hyperbolic systems [27]. He demonstrated that such modifications are determined by the structure of intersection of a curve in the projective space and the Petrovsky hypersurface.

A.A. Tolchennikov investigated the solutions of linear equations of water waves with a localized initial condition describing propagation of long waves (tsunami waves) on the open ocean. Together with S.Yu. Dobrokhotov, V.E. Nazaikinskii, A.I. Shafarevich, and S.Ya. Sekerzh-Zen'kovich, he obtained [28] efficient asymptotic formulas describing the effect on the wave amplitude of different factors: sphericity of the Earth, Coriolis forces arising due to roughness of the bottom and dispersion of focal points.

In addition to that, A.A. Tolchennikov and V.L. Chernyshev [29] investigated the questions associated with propagation of Gauss wave packets on graphs and decorated graphs. They obtained the asymptotic formulas expressing the number of Gauss packets at large times.

5. METRIC GEOMETRY AND GEOMETRIC OPTIMIZATION

Metric geometry and geometric optimization are examined in the works of A.O. Ivanov, A.A. Tuzhilin, and their descendants. The PhD theses of A.O. Ivanov and A.A. Tuzhilin were devoted to the general Plateau problem, and, later, their interest have been focused on the one-dimensional version of this problem—the *generalized Steiner problem*. Here, the boundary is a finite subset M of the metric space (below, MS) X and the "surface" is a connected graph in X whose set of vertices contains M.

A.O. Ivanov and A.A. Tuzhilin created a *theory of locally minimal and extremal networks* (see [30]) and obtained the following results: classification of plane locally minimal binary trees with a convex boundary; description of the local structure of locally minimal networks on Riemannian manifolds and in normed spaces; classification of closed locally minimal networks on closed surfaces of nonnegative curvature (together with I.V. Ptitsyna, Sb. Math. 1992); constraints to the possible topology of plane locally minimal binary tree in terms of the number of convexity levels of the boundary set; estimation of the Steiner ratio for an arbitrary Riemannian manifold; relation between the Steiner ratio of the base and covering space and, as a consequence, determination of Steiner ratios of flat tori and Klein bottles, as well as surfaces of isosceles tetrahedra; derivation of the integral formula of length of the minimal spanning tree connecting no more than countable set of points; uniqueness of the shortest tree on the Euclidean plane for generic boundaries; derivation of the formula for the length of extremal network of fixed topology on a given boundary set generalizing the well-known Maxwell formula.

Of great interest are the so called *hyperspaces*—the MSs whose points are subsets of a fixed space, or all MSs having particular properties. Intensely studied hyperspaces include the *Hausdorff spaces* whose points are bounded closed subsets of a fixed MS and the distance function is the Hausdorff distance, as well as the *Gromov–Hausdorff space* $\mathcal{M}$ whose points are compact MSs and the distance is the Gromov–Hausdorff distance. The latter is defined as the infimum of Hausdorff distances between the images of the pair of MSs at all their isometric inclusions into all possible ambient MSs. The technique of irreducible optimal correspondences proposed by A.O. Ivanov and A.A. Tuzhilin allowed obtaining several significant progress, namely, it was proved that the space $\mathcal{M}$ is geodesic [31] and in this space the Steiner problem is solvable for boundaries consisting of finite spaces (for arbitrary compacts the question is open); the complete proof of triviality of the group of isometries of the space $\mathcal{M}$ was obtained; in terms of the Gromov–Hausdorff distance Ivanov and Tuzhilin solved the generalized Borsuk problem, computed, the chromatic number and the minimal size of the clique covering of an arbitrary graph [32], as well as the smallest dimension of a finite-dimensional MS with the Manhattan metric in which a given finite MS is isometrically included.

Further, in the von Neumann–Bernays–Gödel axiomatics, construction of a proper class $\mathcal{GH}_0$ of all MSs with the Gromov–Hausdorff distance considered up to zero distance was described, and it was shown that the Gromov–Hausdorff distance on $\mathcal{GH}_0$ is a complete internal metric (together with S.I. Borzov, Sb. Math. 2022); it was revealed that multiplication of all distances of an MS X by a given finite number can lead to MS being at an infinite Gromov–Hausdorff distance from X (the multipliers preserving the finiteness of the distance form a group—the stationary group of the space X); it was demonstrated that each cloud, that is, the maximal subclass in $\mathcal{GH}_0$ in which all spaces are at a finite Gromov—Hausdorff distance from each other, contains a single MS invariant with respect to multiplication of all its distances by elements of the stationary group.

At the junction between the theory of geometric optimization problems and metric geometry, A.O. Ivanov and A.A. Tuzhilin constructed a *theory of minimal fillings* (in the Gromov sense) of finite MSs [33]. The filling of a finite MS M is understood as a connected graph G with a weight function ω on edges connecting M. The metric on the vertices of G generated by ω majorizes the original metric on M, and the minimality is understood as the minimality of $\omega(G)$. The weight of the minimal filling of $M \subset X$, considered as an MS with a metric induced from X minorizes the length of the shortest tree in X with the boundary M.

If we fix a tree G and minimize its weight over all weight functions transferring G to a filling of M, then we obtain a problem of linear programming. The set of admissible values of the variables of the dual problem is a convex polyhedron Λ_G depending only on the tree G. The geometry of polyhedra Λ_G is tightly related with the formula for the weight of minimal filling (A.O. Ivanov and A.A. Tuzhilin, 2021).

6. ALGEBRAIC-GEOMETRIC APPROACH TO QUANTUM INTEGRABLE SYSTEMS

The main studies of A.B. Zheglov are devoted to development of the theory of multidimensional Krichever correspondence, whose main task is to generalize the well-known algebraic theory of the Kadomtsev—Petviashvili equation in dimension one. The basic tasks in this theory is the classification of commutative rings of operators met in the theory of ISs (commuting differential, difference, and integrodifferential operators) and study of nonlinear partial differential equations associated with them and their exact solutions.

Since 2002, A.B. Zheglov, together with colleagues from the Steklov Mathematical Institute of the Russian Academy of Sciences, as well as with colleagues from Germany (Humboldt-Universität zu Berlin and Universität zu Köln) D.V. Osipov, G. Kurke, and I. Burban, developed an approach to solving the problem of classification of commuting rings of differential or differential-difference operators, as well as the approach to studying their isospectral deformations based on investigation of spectral sheaves on projective algebraic varieties (the results obtained in this topic in 2002—2016 were presented in the Doctoral Dissertation of A.B. Zheglov "Torsion-free sheaves on manifolds and integrable systems" [4]). Within this approach, principally novel algebraic-geometric and number-theoretic ideas and structures developed by the school of A.N. Parshin are used. In particular, geometric spectral data of a broad class of known examples of algebraically integrable systems were investigated; commutative subalgebras of operators of two variables were classified in terms of geometric spectral data; multidimensional analogs of the Kadomtsev—Petviashvili hierarchy were proposed; and explicit isospectral deformations of known examples of Calogero—Moser algebraically integrable systems with a rational potential were found.

Further studies in this direction are of special interest, because they can lead to solving several problems from various branches of mathematics, including the famous open problems (such as the Jacobian conjecture and the Dixmier conjectures). Here, researchers use knowledge and methods from the algebraic geometry, differential equations, differential algebra, representation theory, K-theory, and other fields. The most recent valuable results of A.B. Zheglov on this topic were presented in [34, 35].

7. ALGEBRAIC GEOMETRY AND MIRROR SYMMETRY

Scientific interests of V.V. Przyjalkowski lie in the field of algebraic geometry and adjacent fields such as of mirror symmetry. Within the classical geometry, he investigated the geometric problems of Fano varieties, mainly of threefolds. For instance, together with I.A. Cheltsov and K.A. Shramov, in the middle of 2000s he classified hyperelliptic and trigonal canonical Gorenstein Fano threefolds. Together with A. Iliev and L. Katsarkov, following the Artin—Mumford approach, he constructed an example of a smooth Fano threefold with torsion in third cohomologies, which, thus, is not stably rational. Another result obtained by V.V. Przyjalkowski and his coauthors is the rationality criterion of a nodal quartic double solid threefold. It generalizes the result of C. Voisin obtained using her breakthrough approach to proving stable non-rationality of varieties. Another direction developed by V.V. Przyjalkowski mainly with I.A. Cheltsov and K.A. Shramov is studying automorphism groups of algebraic varieties. The

[4] http://www.mi.ras.ru/dis/ref16/zheglov/dis.pdf.

classification of infinite groups of automorphisms of smooth Fano threefolds [36] has become one of the most cited results of V.V. Przyjalkowski.

Another important scientific direction of V.V. Przyjalkowski is mirror symmetry. Coming to mathematics from physics, mirror symmetry became to be one of the main mathematical discoveries of recent decades. It established remarkable duality between the algebraic and symplectic geometries, and relates them with arithmetic, analysis, differential geometry, and many other branches of mathematics. V.V. Przyjalkowski developed a theory of toric Landau–Ginzburg models (see review [37]), which enables one to efficiently construct objects that are mirror dual to Fano varieties. Within this approach, he and his coauthors obtained a series of results in this direction, formulated and partially proved several conjectures.

The last direction initiated by application of classical algebraic geometry to the theory of mirror symmetry is the studying weighted complete intersections, that is generalization of usual complete intersections complete intersections (roughly speaking, "general" algebraic varieties). They are one of the main "serial" examples of Fano varieties. Together with K.A. Shramov, V.V. Przyjalkowski obtained several results in this direction; for instance, he established an efficient estimate for the families of smooth weighted complete intersections. In addition to that, he obtained multiple applications of weighted complete intersections. In addition to that, he applied the obtained results to mirror symmetry. V.V. Przyjalkowski has become one of the main experts in this direction in the world. Most general and detailed theory of weighed complete intersections will be outlined in the monograph by V.V. Przyjalkowski and K.A. Shramov.

8. GEOMETRIC TOPOLOGY

E.A. Kudryavtseva with coauthors proved [38] an analog of the Magnus theorem for fundamental groups of surfaces. E.A. Kudryavtseva, S.A. Bogatyi, D.L. Gonçalves, and H. Zieschang solved the Nilsen problem on the minimum number of roots of mappings of a surface in a homotopic class.

E.A. Kudryavtseva proved the realizability criterion of smooth functions on surfaces by height functions at immersions in the three-dimensional Euclidean space. She obtained a combinatorial description of the topology of spaces of smooth functions with given local singularities on surfaces.

9. ALGEBRAIC TOPOLOGY AND ADJACENT TOPICS

F.Yu. Popelenskii together with A.Yu. Onishchenko proved coincidence of spectral sequences for Serre fibration over a compact simply connected manifold beyond the general technology proposed by D. Burns for proving the coincidence of spectral sequences, namely, they considered spectral sequences for a minimal model of fibration $(\Lambda V \otimes \Lambda W, d)$ and the spectral sequences arising from the Čech filtration in the complexes $\check{C}^*(\mathcal{V}, A^*_{PL})$ and $\check{C}^*(\mathcal{V}, S^*(V))$. They constructed an explicit natural isomorphism of these sequences in all terms E_r starting with the second one. It was demonstrated that for a smooth locally trivial fibration the mentioned spectral sequences are isomorphic to the spectral sequences of the complex of smooth forms $\Omega^*(E)$ and the Čech–de Rham complex. Using these results, they computed the rational homologies of the space of free loops of any closed 4-manifold.

Together with D.Yu. Emel'yanov, F.Yu. Popelenskii constructed new series of additive bases in Steenrod algebras $\mod p$, $p > 2$. In addition to the basis of admissible monomials, Milnor basis, and P^t_s bases, they provide a flexible toolkit for working with elements of Steenrod algebras mod p, $p > 2$ and found new applications of these new bases [39]. They proved triviality of the annihilator ideal of the action of the Steenrod algebra in cohomologies of the Eilenberg–MacLane space $K(\mathbb{Z}/p, 2)$. They proved that generalized Toda conjecture for mod $p > 2$ does not hold. Pairs of bases with triangular transituion matrix are found, and the transition matrices are described.

Popelenskii computed cohomology rings with coefficients in $\mathbb{Z}/p$ of partially projective quternionic Stiefel manifolds [40]. He determined the action of any group admitting a free orthogonal action on the sphere S^3 on the Stiefel manifold of orthogonal k-frames in an n-dimensional space: an element of the group is applied to each vector of the frame. The corresponding factor is the partially projective Stiefel manifold.

In the field of dynamical systems, F.Yu. Popelenskii and R.Yu. Pepa investigated discrete Ricci flows for circle packing metrics on triangulated surfaces equipped with additional data (weights) on edges and in vertices. Such flow was proposed by Chow and Luo (2003). The conditions for the weights at which the metric of constant curvature exists and is unique and the convergence of the solution to the Ricci flow equation to this metric takes place for any initial conditions was substantially generalized.

10. THEORY OF KNOTS AND LINKS

The main object studied by D.P. Ilyutko and I.M. Nikonov is diagrams of knots and their applications in the theory of graphs and matroids. The main problem of knot theory is the classification of knots, i.e. embeddings of a closed curve into a three-dimensional sphere—up to an isotopy. The knot is understood as an equivalence class *of diagrams* (of planar 4-valent graphs with a cross structure and over/undercrossing information) under Reidemeister moves [41, 42].

There exists other representations of knot, one of which was constructed by D.P. Ilyutko and A. D. Trofimova. It turns out that any knot has a diagram with a bichromatic coloring, that is, there are two points of the diagram dividing it into two curves such that in each crossing of the diagram both curves are intersected.

Representation of a knot by means of Gauss diagrams and moves on them appeared to be an important idea. Here, not any chord diagram is a Gauss diagram of some (classical) knot. The theory of virtual knots arises if we regard all chord diagrams as Gauss diagrams of some not necessarily embedded 4-valent graphs. Next, we put a Gauss diagram in correspondence with its intersection graph. Rewriting the moves on Gauss diagrams in the language of intersection graphs and considering all simple graphs, we obtain a new object introduced by D.P. Ilyutko and V.O. Manturov—the graph-link. It can be understood as a combinatorial analog of knots in the graph theory. D.P. Ilyutko showed equivalence of two approaches to constructing the theory of graph-links: by means of Gauss diagrams and rotating circuits. Not any simple graph is an intersection graph of a chord diagram, that is, there arises a problem (solved later by V.P. Ilyutko and D.P. Ilyutko) of describing the minimal excluded minors (which make a graph not realizable by a chord diagram) up to transformations.

D.P. Ilyutko and V.O. Manturov generalized several classical invariants of knots to graph-links, constructed new invariants with values in graphs, and performed almost classification for a subclass in free graph-links. They studied the issue of realizability of a graph-link, and, furthermore, D.P. Ilyutko found the first nonrealizable graph-link with many components. The definition of oriented graph-links by D.P. Ilyutko and V.S. Safina provided a generalization of the Jones polynomial to graph-links with several components. I.M. Nikonov determined the Khovanov homology of graph-links.

Framed chord diagrams arise in the theory of finite-order J-invariants of plane curves. D.P. Ilyutko demonstrated incorrectness of the operation of connected sum of chord diagrams modulo framed 4-term relations. D.P. Ilyutko and I.M. Nikonov generalized the Bar-Natan structure of weight systems induced by Lie algebra representations to framed chord diagrams.

11. APPLIED RESEARCH

Applied research in computer geometry began at our chair in 1990s (see the monograph by A. Fomenko and L. Kunii [43]).

In 2007, G.V. Nosovskii, together with D. Liu and O. Sourina, developed [44] a locally adaptive algorithm ADACLUS, which enables one to efficiently cluster point cloud in the Euclidean space in complex situations when the local density of the cloud changes severely, which sophisticates the application of the standard clustering methods. The result was published in a prestigious journal *Pattern Recognition*.

In 2018, G.V. Nosovskii proposed [45] a novel differential-geometric approach to encoding and processing of color images and video sequences based on encoding of the matrix of a digital colored image by means of a special surface $M^2 \subset \mathbb{R}^3$, whose differential-geometric characteristics allow, in particular, rapidly and efficiently (as good as, and sometimes better than, the widely used Canny algorithm) detecting the edges of objects taking into account the color transition, which is important for segmentation and clustering problems. A.Yu. Chekunov, a PhD student of G.V. Nosovskii, implemented an edge detection method applying the CUDA parallel computation architecture. Later, this implementation was supplemented by a postprocessing procedure making the edges thinner to a width of one pixel (2022).

A.O. Ivanov, G.V. Nosovskii, F.Yu. Popelenskii, V.A. Kibkalo, D.A. Fedoseev, their descendants, as well as industrial partners of the chair obtained important results on the manifold conjecture well-known in the data analysis community. They proposed two independent methods for estimating the dimension of a submanifold given by a point cloud close to it in the Euclidean space $\mathbb{R}^N$ for large N, applied to the output set of a neural network (point cloud in $\mathbb{R}^{512}$) in the face recognition problem (https://doi.org/10.48550/arXiv.2107.03903). Novel approaches to anomaly detection for images of different objects and textures see at [46].

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CONFLICT OF INTEREST

The authors of this work declares that they have no conflicts of interest.

REFERENCES

1. A. T. Fomenko, “Variational methods in topology”, in *Works of Outstanding Scientists of the Lomonosov Moscow State University*, 2nd ed. (Moskovskii Gosudarstvennyi Universitet, Moscow, 2023).
2. A. T. Fomenko, *Symplectic Geometry: Methods and Applications*, 2nd ed. (URSS, Lenand, Moscow, 2022).
3. Ya. Ikeda and G. Sharygin, “The argument shift method in universal enveloping algebra $U\mathfrak{gl}_d$”, J. Geom. Phys. **195**, 105030 (2024). https://doi.org/10.1016/j.geomphys.2023.105030
4. G. Sharygin, “Quasiderivations of the algebra $U\mathfrak{gl}_n$ and the quantum Mischenko–Fomenko algebras”, Funct. Anal. Its Appl. **58**, 326–339 (2024). https://doi.org/10.1134/s0016266324030080
5. A. S. Mishchenko and A. T. Fomenko, “Integration of Hamiltonian systems with noncommutative symmetries”, in *Proceedings of the Workshop on Vector and Tensor Analysis* (Moskovskii Gosudarstvennyi Universitet, Moscow, 1981), Vol. 20, pp. 5–54.
6. A. T. Fomenko, “The symplectic topology of completely integrable Hamiltonian systems”, Russ. Math. Surv. **44**, 181–219 (1989). https://doi.org/10.1070/rm1989v044n01abeh002006
7. A. V. Bolsinov and A. T. Fomenko, *Integrable Hamiltonian Systems: Geometry, Topology, Classification* (Udmurtskii Universitet, Izhevsk, 1999), Vols. 1, 2.
8. S. V. Matveev and A. T. Fomenko, “Constant energy surfaces of Hamiltonian systems, enumeration of three-dimensional manifolds in increasing order of complexity, and computation of volumes of closed hyperbolic manifolds”, Russ. Math. Surv. **43**, 3–24 (1988). https://doi.org/10.1070/rm1988v043n01abeh001554
9. A. A. Oshemkov, “Classification of hyperbolic singularities of rank zero of integrable Hamiltonian systems”, Sb. Math. **201**, 1153–1191 (2010). https://doi.org/10.1070/sm2010v201n08abeh004108
10. A. A. Oshemkov, “Saddle singularities of complexity 1 of integrable Hamiltonian systems”, Moscow Univ. Math. Bull. **66**, 60–69 (2011). https://doi.org/10.3103/s0027132211020021
11. I. K. Kozlov and A. A. Oshemkov, “Classification of saddle-focus singularities”, Chebyshevskii Sbornik **21** (2), 228–243 (2020). https://doi.org/10.22405/2226-8383-2020-21-2-228-243
12. A. Bolsinov, L. Guglielmi, and E. Kudryavtseva, “Symplectic invariants for parabolic orbits and cusp singularities of integrable systems”, Philos. Trans. R. Soc., A **376**, 20170424 (2018). https://doi.org/10.1098/rsta.2017.0424
13. V. V. Vedyushkina and I. S. Kharcheva, “Billiard books model all three-dimensional bifurcations of integrable Hamiltonian systems”, Sb. Math. **209**, 1690–1727 (2018). https://doi.org/10.1070/sm9039
14. A. T. Fomenko and V. V. Vedyushkina, “Billiards and integrable systems”, Russ. Math. Surv. **78**, 881–954 (2023). https://doi.org/10.4213/rm10100e
15. V. V. Vedyushkina and V. A. Kibkalo, “Realization of the numerical invariant of the Seifert fibration of integrable systems by billiards”, Moscow Univ. Math. Bull. **75**, 161–168 (2020). https://doi.org/10.3103/s0027132220040075

16. G. V. Belozerov, “Topological classification of billiards bounded by confocal quadrics in three-dimensional Euclidean space”, Sb. Math. **213**, 129–160 (2022). https://doi.org/10.1070/sm9588
17. G. V. Belozerov, “Geodesic flow on an intersection of several confocal quadrics in $\mathbb{R}^n$”, Sb. Math. **214**, 897–918 (2023). https://doi.org/10.4213/sm9864e
18. M. K. Altuev and V. A. Kibkalo, “Topological analysis of pseudo-Euclidean Euler top for special values of the parameters”, Sb. Math. **214**, 334–348 (2023). https://doi.org/10.4213/sm9771e
19. V. A. Kibkalo, “Noncompactness property of fibers and singularities of non-Euclidean Kovalevskaya system on pencil of Lie algebras”, Moscow Univ. Math. Bull. **75**, 263–267 (2020). https://doi.org/10.3103/s0027132220060054
20. A. V. Bolsinov, A. Yu. Konyaev, and V. S. Matveev, “Nijenhuis geometry”, Adv. Math. **394**, 108001 (2022). https://doi.org/10.1016/j.aim.2021.108001
21. A. V. Bolsinov, A. Yu. Konyaev, and V. S. Matveev, “Nijenhuis geometry III: gl-regular Nijenhuis operators”, Rev. Mat. Iberoam. **40**, 155–188 (2024). https://doi.org/10.4171/RMI/1416
22. A. Yu. Konyaev, “Nijenhuis geometry II: Left-symmetric algebras and linearization problem for Nijenhuis operators”, Differ. Geom. Its Appl. **74**, 101706 (2021). https://doi.org/10.1016/j.difgeo.2020.101706
23. A. I. Shafarevich, “Asymptotical and topological constructions in hydrodynamics”, in *Operator Methods in Ordinary and Partial Differential Equations*, Ed. by S. Albeverio, N. Elander, W. N. Everitt, and P. Kurasov, Operator Theory: Advances and Applications, Vol. 132 (Birkhäuser, Basel, 2002), pp. 347–359. https://doi.org/10.1007/978-3-0348-8219-4_25
24. A. I. Esina and A. I. Shafarevich, “Analogs of Bohr–Sommerfeld–Maslov quantization conditions on Riemann surfaces and spectral series of nonself-adjoint operators”, Russ. J. Math. Phys. **20**, 172–181 (2013). https://doi.org/10.1134/s1061920813020052
25. V. L. Chernyshev and A. I. Shafarevich, “Statistics of Gaussian packets on metric and decorated graphs”, Philos. Trans. R. Soc., A **372**, 20130145 (2014). https://doi.org/10.1098/rsta.2013.0145
26. A. I. Allilueva and A. I. Shafarevich, “Delta-type solutions for the non-Hermitian system of induction equations”, Int. J. Theor. Phys. **54**, 3932–3944 (2015). https://doi.org/10.1007/s10773-014-2423-z
27. S. Yu. Dobrokhotov, V. E. Nazaikinskii, and A. I. Shafarevich, “Efficient asymptotics of solutions to the Cauchy problem with localized initial data for linear systems of differential and pseudodifferential equations”, Russ. Math. Surv. **76**, 745–819 (2021). https://doi.org/10.1070/rm9973
28. S. Yu. Dobrokhotov, V. E. Nazaikinskii, and A. A. Tolchennikov, “Uniform formulas for the asymptotic solution of a linear pseudodifferential equation describing water waves generated by a localized source”, Russ. J. Math. Phys. **27**, 185–191 (2020). https://doi.org/10.1134/s1061920820020041
29. V. L. Chernyshev and A. A. Tolchennikov, “The second term in the asymptotics for the number of points moving along a metric graph”, Regular Chaotic Dyn. **22**, 937–948 (2017). https://doi.org/10.1134/s1560354717080032
30. A. O. Ivanov and A. A. Tuzhilin, “Geometry of minimal networks and the one-dimensional Plateau problem”, Russ. Math. Surv. **47**, 59–131 (1992). https://doi.org/10.1070/rm1992v047n02abeh000878
31. A. O. Ivanov, N. K. Nikolaeva, and A. A. Tuzhilin, “The Gromov–Hausdorff metric on the space of compact metric spaces is strictly intrinsic”, Math. Notes **100**, 883–885 (2016). https://doi.org/10.1134/s0001434616110298
32. A. O. Ivanov and A. A. Tuzhilin, “Gromov–Hausdorff distances to simplexes and some applications to discrete optimisation”, Chebyshevskii Sbornik **21** (2), 169–189 (2020). https://doi.org/10.22405/2226-8383-2020-21-2-169-189
33. A. O. Ivanov and A. A. Tuzhilin, “One-dimensional Gromov minimal filling problem”, Sb. Math. **203**, 677–726 (2012). https://doi.org/10.1070/sm2012v203n05abeh004239
34. A. B. Zheglov, “The Schur–Sato theory for quasi-elliptic rings”, Proc. Steklov Inst. Math. **320**, 115–160 (2023). https://doi.org/10.1134/s0081543823010078
35. I. Burban and A. Zheglov, “Cohen–Macaulay modules over the algebra of planar quasi-invariants and Calogero–Moser systems”, Proc. London Math. Soc. **121**, 1033–1082 (2020). https://doi.org/10.1112/plms.12341
36. V. V. Przyjalkowski, I. A. Cheltsov, and K. A. Shramov, “Fano threefolds with infinite automorphism groups”, Izv. Math. **83**, 860–907 (2019). https://doi.org/10.1070/im8834
37. V. V. Przyjalkowski, “Toric Landau–Ginzburg models”, Russ. Math. Surv. **73**, 1033–1118 (2018). https://doi.org/10.1070/rm9852
38. O. Bogopolski, E. Kudryavtseva, and H. Zieschang, “Simple curves on surfaces and an analog of a theorem of Magnus for surface groups”, Math. Z. **247**, 595–609 (2004). https://doi.org/10.1007/s00209-003-0634-8

39. D. Emelyanov and T. Popelensky, “Bases in the mod p Steenrod algebra, β included”, J. Pure Appl. Algebra **223**, 3386–3401 (2019). https://doi.org/10.1016/j.jpaa.2018.11.007
40. G. E. Zhubanov and F. Yu. Popelenskii, “On the cohomology rings of partially projective quaternionic Stiefel manifolds”, Sb. Math. **213**, 300–318 (2022). https://doi.org/10.1070/sm9601
41. V. O. Manturov and D. P. Ilyutko, *Virtual Knots. The State of the Art*, Series on Knots and Everything, Vol. 51 (World Scientific, Singapore, 2013). https://doi.org/10.1142/8438
42. D. P. Ilyutko, V. O. Manturov, and I. M. Nikonov, *Parity and Patterns in Low-Dimensional Topology* (Cambridge Scientific Publishers, Cambridge, 2015).
43. A. Fomenko and L. Kunii, *Topological Modeling for Visualization* (Springer, Tokyo, 1995). https://doi.org/10.1007/978-4-431-66956-2
44. G. V. Nosovskiy, D. Liu, and O. Sourina, “Automatic clustering and boundary detection algorithm based on adaptive influence function”, Pattern Recognit. **41**, 2757–2776 (2008). https://doi.org/10.1016/j.patcog.2008.01.021
45. G. V. Nosovskii, “Geometric encoding of color images”, Moscow Univ. Math. Bull. **73**, 1–8 (2018). https://doi.org/10.3103/s0027132218010011
46. A. O. Ivanov, G. V. Nosovskiy, V. A. Kibkalo, M. A. Nikulin, F. Yu. Popelensky, D. A. Fedoseev, V. Gribushin, V. V. Zlobin, S. S. Kuzin, and I. L. Mazurenko, “Recognition of anomalies of an a priori unknown type”, Chebyshevskii Sbornik **23** (5), 227–240 (2022). https://doi.org/10.22405/2226-8383-2022-23-5-227-240

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