Basic ideas of PDEs geometric theory: an overview

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Introduction

Geometric approach to DEs originates from Sophus Lie's works

- Sophus Lie und Georg Scheffers, *Geometrie der Berührungstransformationen*, Teubner, Leipzig, Band 1 (1891), Band 2 (1893), Band 3 (1896)

His approach in the modern context of differential geometry and homological algebra was developed by A.M. Vinogradov

- A. Vinogradov, *Cohomological Analysis of Partial Differential Equations and Secondary Calculus*, vi+247 pp. Amer. Math. Soc., Providence, RI (2001)

who used the construction of jet spaces

- C. Ehresmann, *Introduction à la théorie des structures infinitésimales et des pseudo-groupes de Lie*. Geometrie Differentielle, Colloq. Inter. du Centre Nat. de la Recherche Scientifique, Strasbourg, 1953, 97–127

Some other references:

References

- A.M. Vinogradov, I.S. Krasil′shchik, V.V. Lychagin, *Introduction to the geometry of nonlinear differential equations*, Nauka, Moscow, 1986. 336 pp (Russian).
- A. Bocharov, V. Chetverikov, S. Duzhin, et al., *Symmetries and Conservation Laws for Differential Equations of Mathematical Physics*, xiv+333 pp. Amer. Math. Soc., Providence, RI (1999). Edited and with a preface by I. Krasil′shchik and A. Vinogradov.
- I. Krasil′shchik and A. Verbovetsky, *Homological methods in equations of mathematical physics*, Open Education & Sciences, Opava, 1998, http://arxiv.org/abs/math/9808130.
- I. Krasil′shchik and A. Vinogradov, *Nonlocal trends in the geometry of differential equations: symmetries, conservation laws, and Bäcklund transformations*, Acta Appl. Math., **15**, pp. 161–209 (1989).

Outline

- Geometry of finite jets
- Equations
- Infinite jets
- Cartan distribution and its symmetries
- Infinite prolongations. Symmetries of DEs
- Vinogradov's spectral sequence
- Conservation laws
- Differential coverings
- Bäcklund transformations
- Tangent and cotangent equations
- Recursion operstors, variations Poisson and symplectic structures

Finite jets

Let M be a smooth ($= C^\infty$) manifold, $\dim M = n$, $\pi \colon E \to M$ be a vector bundle, $\operatorname{rank} \pi = m$. Notation:

- $D(M)$ vector fields
- $\Lambda^i(M)$ differential forms
- $\Gamma(\pi)$ module of (local) sections
- $\gamma(s) \subset E$ graph of $s \in \Gamma(\pi)$

Let $s \in \Gamma(\pi)$ and $[s]_x^k$ denote the class of all $s' \in \Gamma(\pi)$ s.t. $\gamma(s')$ is tangent to $\gamma(s)$ with order k at $\theta = s(x) \in E$, $x \in M$. Jets of order k

$$J^k(\pi) = \{\, [s]_x^k \mid x \in M,\ s \in \Gamma(\pi) \,\}$$

In particular, $J^0(\pi) = E$
Natural maps:

$$\pi_k \colon J^k(\pi) \to M, \qquad [s]_x^k \mapsto x,$$
$$\pi_{k,l} \colon J^k(\pi) \to J^l(\pi), \qquad [s]_x^k \mapsto [s]_x^l,\ k \geq l$$

Finite jets

Let $\mathscr{U} \subset M$ and $x^1, \dots, x^n$ be local coordinates. Assume that $\pi^{-1}(\mathscr{U}) = M \times \mathbb{R}^m$ is a trivialization with coordinates $u^1, \dots, u^m$ in $\mathbb{R}^m$. Then the functions u^j_σ, $|\sigma| \le k$,

$$u^j_\sigma(\theta) = \left. \frac{\partial^{|\sigma|} s^j}{\partial x^\sigma} \right|_x, \theta = [s]^k_x, \quad x \in M, \quad s \in \Gamma(\pi),$$

are local coordinates in $\pi^{-1}(\mathscr{U})$; they are called *adapted*. Thus:

- $J^k(\pi)$ is a smooth manifold
- π_k, $\pi_{k,l}$ are smooth bundles
- the maps $j_k(s) \colon M \to J^k(\pi)$, $x \mapsto [s]^k_x$ (jets of sections) are smooth sections of π_k
- $\dim J^k(\pi) = n + mN$, where N is the number of partial derivatives of order $\le k$

Finite jets

A point $\theta = [s]_x^k$ may be understood as a pair $(\theta', R_{\theta'})$, where $\theta' = \pi_{k,k-1}(\theta) \in J^{k-1}(\pi)$ and $R_{\theta'}$ is the tangent plane the graph of $j_{k-1}(s)$.

- The space $\mathscr{C}_\theta = \pi_{k,k-1,*}^{-1}(R_{\theta'}) \subset T_\theta(J^k(\pi))$ is the Cartan plane.

The correspondence $\mathscr{C}\colon \theta \mapsto \mathscr{C}_\theta$ is the Cartan distribution.

Geometry of jets = geometry of the pair $(J^k(\pi), \mathscr{C})$

- A Lie–Bäcklund transformation is a diffeomorphism

$F\colon J^k(\pi) \to J^k(\pi)$ that preserves $\mathscr{C}$, i.e.

$$F_*(\mathscr{C}_\theta) = \mathscr{C}_{F(\theta)}.$$

Main properties of these transformations, as well as of $\mathscr{C}$ itself are discussed below.

Finite jets

Properties of the Cartan distribution

- it is "maximally nonintegrable", i.e., the derived distribution $\mathscr{C}^{(r)}$ coincides with the entire $TJ^k(\pi)$ for some $r > 0$
- $\operatorname{rank} \mathscr{C} = n + mN$, where N is the number of partial derivatives of order k
- $\mathscr{C}_\theta$ is the span of tangent planes to graphs of jets passing through the point θ
- in adapted coordinates, $\mathscr{C}$ is defined by one-forms

  $$\omega^j_\sigma = du^j_\sigma - \sum_i u^j_{\sigma i} dx^i, \qquad |\sigma| \leq k-1,$$

  the Cartan forms
- if $\operatorname{rank} \pi = 1$ then $J^1(\pi)$ is a contact manifold

Finite jets

Let $\Phi\colon J^k(\pi) \to J^k(\pi)$ be a L–B transformation. Its lift to $J^{k+1}(\pi)$ is defined. For a point $\theta = (\theta', R_{\theta'}) \in J^{k+1}(\pi)$, $\theta' \in J^k(\pi)$,

$$\Phi^{(1)}(\theta) = (\Phi(\theta'), \Phi_*(R_{\theta'}))$$

The lift is a L–B transformation again and one can define $\Phi^{(l)}\colon J^{k+l}(\pi) \to J^{k+l}(\pi)$ for any $l \geq 0$. They are in agreement:

$$\begin{array}{ccc} J^{k+r}(\pi) & \xrightarrow{\Phi^{k+r}} & J^{k+r}(\pi) \\ {\scriptstyle \pi_{k+r,k+l}}\downarrow & & \downarrow{\scriptstyle \pi_{k+r,k+l}} \\ J^{k+l}(\pi) & \xrightarrow{\Phi^{k+l}} & J^{k+l}(\pi) \end{array}$$

for all $r \geq l \geq 0$.

Finite jets

Lie–Bäcklund Theorem: Let $\Phi\colon J^k(\pi) \to J^k(\pi)$ be a Lie–Bäcklund transformation. Then

- $\Phi = G^{(k)}$ for some diffeomorphism of $J^0(\pi) = E$ if $\operatorname{rank}\pi > 1$,
- $\Phi = G^{(k-1)}$ for some contact transformation of $J^1(\pi)$ when $\operatorname{rank}\pi = 1$.

!!! Lie–Bäcklund transformations do not preserve the bundle structure in $\pi\colon E \to M$

Generalization: jets of submanifolds

Equations

A differential equation of order k (imposed on sections of π) is a submanifold $\mathscr{E} \subset J^k(\pi)$. One can always assume that $\mathscr{E} = \{F = 0\}$, where F is a section of some bundle ξ over $J^k(\pi)$.
The Cartan distribution on $\mathscr{E}$ is

$$\mathscr{C} = \mathscr{C}_{\mathscr{E}} \colon \theta \in \mathscr{E} \mapsto T_\theta \mathscr{E} \cap \mathscr{C}_\theta$$

Solutions of $\mathscr{E}$ are π_k-horizontal n-dimensional integral manifolds.

- A morphism $\Phi \colon \mathscr{E} \to \mathscr{E}'$ of equations is a smooth map s.t. $\Phi_*(\mathscr{C}_\theta) \subset \mathscr{C}'_{\Phi(\theta)}$
- equations are equivalent if Φ is a diffeomorphism
- an isomorphism $\Phi \colon \mathscr{E} \to \mathscr{E}$ is a (finite classical) symmetry. It takes solutions to solutions
- an infinitesimal symmetry is a vector field whose one-parametric group consists of finite symmetries

!!! These “interior” definitions are impossible to implement practically. Alternative: “exterior” ones

Equations

Examples

- KdV equation

$$u_t = uu_x + u_{xxx}$$

  is an equation in $J^3(\pi)$ for the trivial bundle $\pi\colon \mathbb{R}\times\mathbb{R}^2 \to \mathbb{R}^2$
- Maxwell equation $\{d\omega = 0,\ d*\omega = 0\}$ is a linear equation in $J^1(\pi)$ for $\pi\colon \wedge^2 T^*M \to M$.
- Consider the subbundle π in $TM \otimes T^*M = \mathrm{End}(TM) \to M$ given by the condition $A^2 = -\mathrm{id}$. Then $\{[\![A,A]\!]_{\mathbf{n}} = 0\} \subset J^1(\pi)$, where $[\![A,A]\!]_{\mathbf{n}}$ is the Nijenhuis bracket, is the equation of complex structures. Here π in *not* a vector bundle.
- Let $\pi\colon \wedge^2 TM \to M$ be the bundle of bi-vectors. Then $\{[\![P,P]\!]_{\mathbf{s}} = 0\} \subset J^1(\pi)$, where $[\![P,P]\!]_{\mathbf{s}}$ is the Schouten bracket, is the equation of Poisson structures.

Examples

- Let $\xi : E \to M$ be a bundle and $\pi = \xi_{1,0} : J^1(\xi) \to E$. Sections of π are connections in ξ. The equation $\{R_\nabla = 0\} \subset J^1(\pi)$, where R_∇ is the curvature of $\nabla \in \Gamma(\pi)$, defines flat connections. Here π is *not* a vector bundle.
- The sine-Gordon equation $u_{xy} = \sin u$ and the equation of minimal surface $(1+u_x^2)u_{yy} - 2u_x u_y u_{xy} + (1+u_y^2)u_{xx} = 0$ *look* like equations in 2-jets of $\mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$. But actually they are equations in J^2 of 2D submanifolds in $\mathbb{R}^3$.

Applications of symmetries

Solitons: The field $v\partial/\partial x + \partial/\partial t$, $v = \text{const}$, is an obvious symmetry of KdV $u_t = uu_x + u_{xxx}$. Its invariant solutions (travelling waves) are 1-solitons of the KdV.

Lie–Bianchi Theorem: Let $\mathscr{E}$ be a scalar ODE of order k, i.e., $\dim M = 1$, $\operatorname{rank} \pi = 1$. Then if $\mathscr{E}$ admits a solvable k-dimensional Lie algebra of infinitesimal symmetries, then it is integrable in quadratures.

Infinite jets

Informally, $J^\infty(\pi)$ is the space of Taylor series of sections $s \in \Gamma(\pi)$. More exactly, the space of infinite jets is the inverse limit of

$$\ldots \longrightarrow J^{k+1}(\pi) \xrightarrow{\pi_{k+1,k}} J^k(\pi) \longrightarrow \ldots \longrightarrow J^1(\pi) \xrightarrow{\pi_{1,0}} E \xrightarrow{\pi} M$$

Its points are classes $[s]_x^\infty$ of sections with infinite order of tangency.

Since $\dim J^\infty(\pi) = \infty$, we shall stick to the following conventions:

- The algebra of smooth functions on $J^\infty(\pi)$ is the filtered algebra

$$\mathscr{F}(\pi) = \bigcup_{k=0}^{\infty} C^\infty(J^k(\pi))$$

- The same for the modules $\Lambda^i(\pi)$ of differential forms
- The module $D(\pi)$ of vector fields consists of $\mathbb{R}$-linear derivations of $\mathscr{F}(\pi)$

!!! Generically, vector fields on J^∞ do not possess trajectories

Cartan distribution and connection

Let $\theta = [s]_x^\infty \in J^\infty(\pi)$. All graphs of infinite jets $j_\infty(s')$ passing through θ are tangent to each other. Their common tangent plane $\mathscr{C}_\theta$ is the Cartan plane, while

$$\mathscr{C} : J^\infty(\pi) \ni \theta \mapsto \mathscr{C}_\theta \subset TJ^\infty(\pi)$$

is the Cartan distribution. One has

- This distribution is horizontal and $\operatorname{rank} \mathscr{C} = n$. Hence, it defines a connection $\mathscr{C} : D(M) \to D(\pi)$, the Cartan connection
- The Cartan distribution is *formally* integrable, i.e., $[X, Y] \in \mathscr{C}$ whenever X, $Y \in \mathscr{C}$. Consequently, the connection is flat, i.e., $\mathscr{C}([X,Y]) = [\mathscr{C}(X), \mathscr{C}(Y)]$.
- Graphs of infinite jets, and them only, are maximal integral manifolds of $\mathscr{C}$.

Evolutionary vector fields

Let $S(\pi) = \{X \in D(\pi) \mid [X, \mathscr{C}] \subset \mathscr{C}\}$. Then $S(\pi)$ is a Lie algebra and $\mathscr{C} \subset S(\pi)$ is its ideal. We set $\mathrm{sym}(\pi) = S(\pi)/\mathscr{C}$. So a symmetry is a coset. But due to properties of $\mathscr{C}$,

$$D(\pi) = D^v(\pi) \oplus \mathscr{C},$$

where $D^v(\pi)$ is the Lie algebra of π_∞-vertical fields and thus

$$\mathrm{sym}(\pi) \simeq \{X \in D^v(\pi) \mid [X, \mathscr{C}] \subset \mathscr{C}\}$$

Theorem: There is a one-to-one correspondence between $\mathrm{sym}(\pi)$ and $\varkappa(\pi) = \Gamma(\pi_\infty^*(\pi))$.

The vector field $\mathbf{E}_\varphi$ corresponding to $\varphi \in \varkappa(\pi)$ is the *evolutionary field* with the *generating section* φ. Lie algebra structure in $\varkappa(\pi)$:

$$[\mathbf{E}_{\varphi_1}, \mathbf{E}_{\varphi_2}] = \mathbf{E}_{\{\varphi_1, \varphi_2\}};$$

$\{\varphi_1, \varphi_2\}$ is the Jacobi bracket.

Coordinates

Let x^i, u^j_σ, $i = 1, \dots, n$, $j = 1, \dots, m$, $|\sigma| \geq 0$, be adapted coordinates. Then $\mathscr{C}$ annihilates the infinite system of *Cartan forms*

$$\omega^j_\sigma = du^j_\sigma - \sum_i u^j_{\sigma i} dx^i$$

and is spanned by the *total derivatives*

$$D_i = \mathscr{C}(\partial/\partial x^i) = \frac{\partial}{\partial x^i} + \sum_{j,\sigma} u^j_{\sigma i} \frac{\partial}{\partial u^j_\sigma}$$

For $\varphi = (\varphi^1, \dots, \varphi^m) \in \varkappa(\pi)$,

$$\mathbf{E}_\varphi = \sum_{j,\sigma} D_\sigma(\varphi^j) \frac{\partial}{\partial u^j_\sigma},$$

where D_σ is the composition of total derivatives corresponding to σ, and

$$\{\varphi_1, \varphi_2\}^j = \sum_{\alpha,\sigma} \left(D_\sigma(\varphi_1^\alpha) \frac{\partial \varphi_2^j}{\partial u^\alpha_\sigma} - D_\sigma(\varphi_2^\alpha) \frac{\partial \varphi_1^j}{\partial u^\alpha_\sigma} \right)$$

Variational bi-complex

The splitting $D(\pi) = D^v(\pi) \oplus \mathscr{C}$ induces the dual

$$\Lambda^1(\pi) = \Lambda^1_h(\pi) \oplus \Lambda^1_v(\pi),$$

where

$$\Lambda^1_h(\pi) = \{\sum a_i\,dx^i \mid a_i \in \mathscr{F}(\pi)\}, \quad \Lambda^1_v(\pi) = \{\sum b_{j,\sigma}\omega^j_\sigma \mid b_{j,\sigma} \in \mathscr{F}(\pi)\}$$

are *horizontal* and Cartan (*vertical*) forms, resp. Set $\Lambda^{p,q}(\pi) = \Lambda^q_h(\pi) \wedge \Lambda^p_v(\pi)$. Then $d = d_h + d_v$, where

$$d_h(f) = \sum D_i(f)\,dx^i, \quad d_v(f) = \sum \frac{\partial f}{\partial u^j_\sigma}\omega^j_\sigma$$

and

$$d_h \circ d_v + d_v \circ d_h = 0, \quad d_h^2 = d_v^2 = 0.$$

The corresponding spectral sequence is a particular case of Vinogradov' spectral sequence and converges to the de Rham cohomology $H^*_{\mathrm{dR}}(E)$ of $J^0(\pi)$.

Variational bi-complex

The term $\mathscr{C}E_0^{p,q}$ is

$$
\begin{array}{cccc}
\Lambda_h^n & \dots & \Lambda^{p,n} & \dots \\
\vdots & & \vdots & \\
\Lambda_h^{q+1} & \dots & \Lambda^{p,q+1} & \dots \\
d_h \uparrow & & d_h \big\uparrow & \\
\Lambda_h^q & \dots & \Lambda^{p,q} & \dots \\
\vdots & & \vdots & \\
\mathscr{F} & \dots & \Lambda_v^p & \dots
\end{array}
$$

To describe $\mathscr{C}E_1^{p,q}$ additional constructions are needed.

$\mathscr{C}$-differential operators

- Notation: Let $\xi : B \to M$ be a vector bundle; denote $\mathscr{F}(\pi,\xi) = \Gamma(\pi^*(\xi))$. Elements of $\mathscr{F}(\pi,\xi)$ are nonlinear DOs $\Gamma(\pi) \to \Gamma(\xi)$. E.g., $\varkappa(\pi) = \mathscr{F}(\pi,\pi)$, $\Lambda^q_h(\pi) = \mathscr{F}(\pi,\tau^q)$, where $\tau^q\colon\ \wedge^q T^*M \to M$.

A linear DO $\Delta\colon\ \mathscr{F}(\pi,\xi) \to \mathscr{F}(\pi,\eta)$ is a $\mathscr{C}$-differential (or total) operator if it admits restrictions to graphs of infinite jets, i.e., for any $s \in \Gamma(\pi)$ there exists $\Delta_s\colon\ \Gamma(\xi) \to \Gamma(\eta)$, s.t.

$$j_\infty(s)^* \circ \Delta = \Delta_s \circ j_\infty(s)^*.$$

Locally this means that in adapted coordinates Δ is an operator in total derivatives.

$\mathscr{C}$-differential operators

Two important examples

- For a linear DO $A\colon \Gamma(\xi) \to \Gamma(\eta)$ one can construct the operator $\mathscr{C}_A\colon \mathscr{F}(\pi,\xi) \to \mathscr{F}(\pi,\eta)$. In particular, $d_h\colon \Lambda_h^*(\pi) \to \Lambda_h^*(\pi)$ is $\mathscr{C}_d$ for the de Rham differential $d\colon \Lambda^*(M) \to \Lambda^*(M)$.
- Any evolutionary vector field $\mathbf{E}_\varphi\colon \mathscr{F}(\pi) \to \mathscr{F}(\pi)$ admits a natural extension $\mathbf{E}_\varphi^\xi\colon \mathscr{F}(\pi,\xi) \to \mathscr{F}(\pi,\xi)$, s.t.

$$\mathbf{E}_\varphi^\xi(fF) = \mathbf{E}_\varphi(f)F + f\,\mathbf{E}_\varphi^\xi(F), \quad f \in \mathscr{F}(\pi),\ F \in \mathscr{F}(\pi,\xi).$$

Define linearization $\ell_F\colon \varkappa(\pi) \to \mathscr{F}(\pi,\xi)$ of F by

$$\ell_F(\varphi) = \mathbf{E}_\varphi^\xi(F), \quad F \in \mathscr{F}(\pi,\xi).$$

In adapted coordinates, ℓ_F is a matrix operator with the entries

$$(\ell_F)^\alpha_\beta = \sum_\sigma \frac{\partial F^\alpha}{\partial u^\beta_\sigma} D_\sigma$$

for $F = (F^1, \dots, F^\alpha, \dots)$

Horizontal de Rham complex and adjoint operators

The 1^{st} column of $\mathscr{C}E_0$ is

$$0 \longrightarrow \mathscr{F} \xrightarrow{d_h} \Lambda_h^1 \xrightarrow{d_h} \ldots \xrightarrow{d_h} \Lambda_h^q \xrightarrow{d_h} \Lambda_h^{q+1} \xrightarrow{d_h} \ldots$$

and is called the horizontal de Rham complex. Its cohomologies are denoted by $H_h^q(\pi)$.
For any two modules $P = \mathscr{F}(\pi,\xi)$, $\mathscr{F}(\pi,\eta)$ denote by $\mathscr{C}\mathrm{Diff}(P,Q)$ the module of $\mathscr{C}$-differential operators. Then the complex

$$\ldots \longrightarrow \mathscr{C}\mathrm{Diff}(P,\Lambda_h^q) \xrightarrow{d_h^P} \mathscr{C}\mathrm{Diff}(P,\Lambda_h^{q+1}) \longrightarrow \ldots \longrightarrow \mathscr{C}\mathrm{Diff}(P,\Lambda_h^n) \longrightarrow 0,$$

where $d_h^P(A) = d_h \circ A$, is acyclic in all terms but the last one.

Horizontal de Rham complex and adjoint operators

Its cohomology $H^n(d_h^P) = \hom_{\mathscr{F}(\pi)}(P, \Lambda_n^n(\pi)) = \hat{P}$. So, for any $\Delta\colon P \to Q$ its *adjoint* $\Delta^*\colon \hat{Q} \to \hat{P}$ is defined. It is also a $\mathscr{C}$-differential DO and enjoys the Green formula

$$\langle \Delta(p), \hat{q} \rangle - \langle p, \Delta^*(\hat{q}) \rangle = d_h \omega_{p,\hat{q}}, \quad \omega_{p,q} \in \Lambda_h^{n-1}(\pi),\ p \in P, \hat{q} \in \hat{Q},$$

while the correspondence $(p, \hat{q}) \mapsto \omega_{p,\hat{q}}$ is a $\mathscr{C}$-differential DO in both arguments.

If $\Delta = \sum_\sigma a_\sigma D_\sigma$, then $\Delta^* = \sum_\sigma (-1)^{|\sigma|} D_\sigma \circ a_\sigma$. For a matrix operator $\Delta = (\Delta_{i,j})$, one has $\Delta^* = (\Delta_{i,j}^*)^{\mathrm{t}}$.

Variational bi-complex. The term $\mathscr{C}E_1^{p,q}$

The term is of the form

$$\begin{array}{cccc} H_h^n(\pi) \xrightarrow{\delta_0} & K_0(\varkappa) \xrightarrow{\delta_1} & K_1(\varkappa) \rightarrow & \dots \\ H_h^{n-1}(\pi) & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \\ H_h^0(\pi) & 0 & 0 & \dots \end{array}$$

where $K_p(\varkappa)$ consists of the maps $\underbrace{\varkappa \otimes \dots \otimes \varkappa}_{p \text{ times}} \rightarrow \hat{\varkappa}$, which are

1. Skew-symmetric.
2. Anti-self-adjoint $\mathscr{C}$-differential operators in each argument.

The top line is exact.

Variational bi-complex. Lagrangian formalism

The groups $H_h^q(\pi)$ coincide with $H_{\mathrm{dR}}^q(J^0(\pi))$ for all $0 \le q \le n-1$. If the latter vanishes, we obtain the exact sequence

$$\mathscr{F}(\pi) \xrightarrow{d_h} \Lambda_h^1 \longrightarrow \dots \longrightarrow \Lambda_h^{n-1} \xrightarrow{d_h} \Lambda_h^n \xrightarrow{\delta} K_0(\varkappa) \xrightarrow{\delta_1} K_1(\varkappa) \longrightarrow \dots$$

which is called the global variational bi-complex. The following facts deserve attention:

1. Elements $\mathscr{L} = L\,dx^1 \wedge \cdots \wedge dx^n$ of $\Lambda_h^n(\pi)$ may be understood as Lagrangian densities.
2. $\delta(\mathscr{L}) = \ell_L^*(1) = \left(\frac{\delta L}{\delta u^1}, \dots, \frac{\delta L}{\delta u^m}\right)$, where $\delta/\delta u^j$ is the variational derivative, i.e., δ is the Euler operator.
3. $\delta_1(\psi) = \ell_\psi - \ell_\psi^*$.

Variational bi-complex. Lagrangian formalism

From exactness of the variational bi-complex it follows that

1. $\ker \delta = \operatorname{im} d_h$, i.e., variationally trivial Lagrangians are total divergences.
2. $d_h \omega = 0$ if and only if ω is of the form $u = d_h \rho$, $\rho \in \Lambda_h^{n-1}(\pi)$, i.e., null total divergences are total curls.
3. $\psi = \delta(\mathscr{L})$, i.e., ψ is an Euler–Lagrange operator, if and only if $\ell_\psi = \ell_\psi^*$, and this solves the inverse problem of the calculus of variations.

Infinite prolongations. A motivating example

KdV equation $u_t = uu_x + u_{xxx}$ is a classical example of integrable systems:

- There exists infinite number of "flows" $u_t = \varphi_{2k+1}(u, u_x, \dots)$ of *unlimited jet order* commuting with KdV.
- They are obtained from $u_t = u_x$ by means of the operator

$$\mathscr{R} = D_x^2 + \frac{2}{3}u + \frac{1}{3}u_x D_x^{-1}$$

- There exists infinite number of "conserved densities" X_{2k} also of unlimited jet order.
- KdV may be represented as

$$u_t = D_x \frac{\delta X_1}{\delta u} = (D_x^3 + \frac{2}{3}uD_x + \frac{1}{3}u_x)\frac{\delta X_2}{\delta u}$$

  and this allows to define two Poisson brackets on the space of conserved densities.
- These brackets are compatible, while the densities are in involution w.r.t. both ones

Infinite prolongations

Let $\mathscr{E} = \{F = 0\} \subset J^k(\pi)$, $F \in P = \mathscr{F}(\pi, \xi)$. The 1^{st} prolongation of $\mathscr{E}$ is

$$\mathscr{E}^{(1)} = \{\, \theta = (\theta', R_{\theta'}) \in J^{k+1}(\pi) \mid R_{\theta'} \subset T_\theta \mathscr{E} \,\} \subset J^{k+1}(\pi).$$

By induction, $\mathscr{E}^{(k+1)} = \left(\mathscr{E}^{(k)}\right)^{(1)}$ whenever it is possible. Behavior of prolongations is governed by Cartan's involutivity theory and Spencer cohomology (H. Goldschmidt).

We stick to the formally integrable case: all $\mathscr{E}^{(k)}$ are smooth manifolds and $\mathscr{E}^{(k+1)} \to \mathscr{E}^{(k)}$ are smooth bundles. Then the inverse limit $\mathscr{E}^{(\infty)} \subset J^\infty(\pi)$ is called the infinite prolongation of $\mathscr{E}$.

!!! From now on we work with infinite prolongations only and $\mathscr{E}$ will stand for $\mathscr{E}^{(\infty)}$

Infinite prolongations

We assume:

- Smooth functions $\mathscr{F}(\mathscr{E}) = \mathscr{F}(\pi)\big|_{\mathscr{E}}$
- Differential forms $\Lambda^i(\mathscr{E}) = \Lambda^i(\pi)\big|_{\mathscr{E}}$
- $\varkappa(\mathscr{E}) = \varkappa(\pi)|_{\mathscr{E}}$
- Vector fields $D(\mathscr{E})$ are $\mathbb{R}$-linear derivations $\mathscr{F}(\mathscr{E}) \to \mathscr{F}(\mathscr{E})$
- The Cartan distribution $\mathscr{C} = \mathscr{C}_{\mathscr{E}} \colon \mathscr{E} \ni \theta \mapsto \mathscr{C}_\theta \cap T_\theta \mathscr{E} \subset T_\theta \mathscr{E}$

We also assume that

- $\pi_{\infty,0}(\mathscr{E})$ is everywhere dense in $J^0(\pi)$, i.e., no 0-order relations
- If $\square$ is a covariant object on $J^\infty(\pi)$ and $\square|_{\mathscr{E}} = 0$, then $\square = \Delta(F)$,

where Δ is a $\mathscr{C}$-differential DO

Two facts are worth to mention

- In adapted coordinates, infinite prolongation is defined by the infinite system

$$D_\sigma(F) = 0, \qquad |\sigma| \geq 0$$

- $\mathscr{C}$-differential DOs admit restriction from $J^\infty(\pi)$ to $\mathscr{E}$

Remark To restrict to $\mathscr{E}$ amounts to rewrite in *internal coordinates*

Symmetries

A (higher infinitesimal, or generalized) symmetry of $\mathscr{E}$ is a π_∞-vertical vector field S, s.t. $[S,\mathscr{C}] \subset \mathscr{C}$

Theorem: Any symmetry of $\mathscr{E}$ is of the form $\mathbf{E}_\varphi$, where the generating section $\varphi \in \varkappa(\mathscr{E})$ satisfies

$$\ell_{\mathscr{E}}(\varphi) = 0$$

and $\ell_{\mathscr{E}} = \ell_F|_{\mathscr{E}}$.

Symmetries form a Lie algebra $\operatorname{sym}\mathscr{E}$ w.r.t. commutator.

Below, as a rule, we identify a symmetry $\mathbf{E}_\varphi$ with its generating section φ. The Lie operation in the space of sections is the Jacobi bracket $\{\cdot,\cdot\}$

Example: KdV

It is convenient to choose the functions x, t, $u_k = u_{\underbrace{x\ldots x}_{k\text{ times}}}$, $k \geq 0$, for internal coordinates in $\mathscr{E}$ for KdV (as well as for any other evolutionary equation). The restrictions of total derivatives in these coordinates are

$$D_x = \frac{\partial}{\partial x} + \sum_k u_{k+1} \frac{\partial}{\partial u_k}, \quad D_t = \frac{\partial}{\partial t} + \sum_k D_x^k(uu_1 + u_3) \frac{\partial}{\partial u_k}.$$

We also have

$$\ell_{\mathscr{E}} = D_t - u_x - uD_x - D_x^3$$

Solving the equation $\ell_{\mathscr{E}}(\varphi) = 0$ up to order 7 provides the following solutions

Example: KdV

$$\varphi_1 = u_1, \qquad \varphi_3 = u_5 + \frac{5}{3}uu_3 + \frac{10}{3}u_1u_2 + \frac{5}{6}u^2u_1,$$

$$\varphi_2 = u_3 + uu_1, \qquad \varphi_4 = u_7 + \frac{7}{3}uu_5 + 7u_1u_4 + \dots,$$

$$\varphi_5 = tu_1 + 1, \qquad \varphi_6 = tu_3 + \frac{1}{3}(3tu + x)u_1 + \frac{2}{3}u.$$

Symmetries φ_5, φ_6 constitute a 2-dimensional solvable Lie algebra:

$$[\varphi_5, \varphi_6] = \frac{2}{3}\varphi_5,$$

which is represented in the Abelian ideal formed by $\varphi_1, \dots, \varphi_4$:

$$\{\varphi_5, \varphi_1\} = 0, \qquad \{\varphi_6, \varphi_1\} = \frac{1}{3}\varphi_1,$$

$$\{\varphi_5, \varphi_2\} = \varphi_1, \qquad \{\varphi_6, \varphi_2\} = \varphi_2,$$

$$\{\varphi_5, \varphi_3\} = \frac{5}{3}\varphi_2, \qquad \{\varphi_6, \varphi_3\} = \frac{5}{3}\varphi_3,$$

$$\{\varphi_5, \varphi_4\} = \frac{7}{3}\varphi_3, \qquad \{\varphi_6, \varphi_4\} = \frac{7}{3}\varphi_4$$

Example: KdV

Remarks:

1. Symmetries $\varphi_1, \ldots, \varphi_4$ are the 1^{st} four flows that commute with KdV (φ_3 is KdV itself)
2. The classical theory of symmetries does not "see" the symmetries φ_3 and φ_4

Vinogradov's spectral sequence

Let $\mathscr{E} \subset J^\infty$ and $\Lambda^1_v(\mathscr{E})$ be the module of forms vanishing on $\mathscr{C}$. Then, since $\mathscr{C}$ is integrable, the ideal $\mathscr{C}^1\Lambda(\pi) \subset \Lambda^*(\pi)$ is closed w.r.t. d. Consequently, the filtration

$$\Lambda^*(\mathscr{E}) \supset \mathscr{C}^1\Lambda(\pi) \supset \cdots \supset \mathscr{C}^i\Lambda(\pi) \supset \mathscr{C}^{i+1}\Lambda(\pi) \supset \ldots$$

by the powers of this ideal defines a spectral sequence that converges to the de Rham cohomology $H^*_{\mathrm{dR}}(\mathscr{E})$. This is the Vinogradov $\mathscr{C}$-spectral sequence.

Remark: The bi-complex discussed above is a particular case of this system when $\mathscr{E}$ is an equation in jets of a vector bundle.

Term E_0

$$
\begin{array}{|cccc|}
\hline
E_0^{0,n} & \cdots & E_0^{p,n} & \cdots \\
\vdots & & \vdots & \\
E_0^{0,q+1} & \cdots & E_0^{p,q+1} & \cdots \\
d_0 \uparrow & & d_0 \uparrow & \\
E_0^{0,q} & \cdots & E_0^{p,q} & \cdots \\
\vdots & & \vdots & \\
\mathscr{F}(\mathscr{E}) & \cdots & E_0^{0,p} & \cdots \\
\hline
\end{array}
$$

Here $E_0^{p,q}(\mathscr{E}) = \mathscr{C}^p\Lambda^{p+q}(\mathscr{E})/\mathscr{C}^{p+q}\Lambda(\mathscr{E})$.

Term E_1

$$
\begin{array}{ccccccccccc}
E_1^{0,n} & \xrightarrow{d_1} & E_1^{1,n} & \xrightarrow{d_1} & \dots & \xrightarrow{d_1} & E_1^{p,n} & \xrightarrow{d_1} & E_1^{p+1,n} & \xrightarrow{d_1} & \dots \\
\vdots & & \vdots & & & & \vdots & & \vdots & & \\
E_1^{0,n-q+1} & \xrightarrow{d_1} & E_1^{1,n-q+1} & \xrightarrow{d_1} & \dots & \xrightarrow{d_1} & E_1^{p,n-q+1} & \xrightarrow{d_1} & E_1^{p+1,n-q+1} & \xrightarrow{d_1} & \dots \\
E_1^{0,n-q} & \xrightarrow{d_1} & 0 & & & & 0 & & 0 & & \\
\vdots & & \vdots & & & & \vdots & & \vdots & & \\
E_1^{0,0} & \xrightarrow{d_1} & 0 & & & & 0 & & 0 & &
\end{array}
$$

Term E_1

If $H^*_{\mathrm{dR}}(\mathscr{E})$ is trivial, the terms $E_1^{0,1}, \dots, E_1^{0,n-q}$ vanish. The number q of the 1st from the top nontrivial lines in E_1 is determined by the length of the *compatibility complex*

$$0 \longrightarrow \varkappa \xrightarrow{\ell_{\mathscr{E}}=\Delta_0} P = P_0 \xrightarrow{\Delta_1} P_1 \xrightarrow{\Delta_2} P_2 \longrightarrow \dots \xrightarrow{\Delta_N} P_N \longrightarrow 0$$

Here Δ_{i+1} is the compatibility operator for Δ_i, i.e., the minimal operator, s.t. $\Delta_{i+1} \circ \Delta_i = 0$. Then $q = N + 2$.

The simplest case is 2-line equations. All scalar equations are of this kind, as well as the majority of equations of mathematical physics.

Nevertheless, for equations in field theory (Maxwell, Einstein, Yang–Mills) $q > 2$. E.g., $q = n + 1$ for $\ell_d = d_h$ (the simplest version of Maxwell).

Conservation laws and cosymmetries

Let us stick to the 2-line case. Note that

$$E_0^{0,q}(\mathscr{E}) = \Lambda^q(\mathscr{E})/\mathscr{C}^1\Lambda^q(\mathscr{E}) = \Lambda_h^q(\mathscr{E}).$$

So, 0-column of E_1

$$0 \longrightarrow \mathscr{F}(\mathscr{E}) \xrightarrow{d_h} \Lambda_h^1(\mathscr{E}) \xrightarrow{d_h} \dots \xrightarrow{d_h} \Lambda_h^{n-1}(\mathscr{E}) \xrightarrow{d_h} \Lambda_h^n(\mathscr{E}) \longrightarrow 0$$

is the horizontal de Rham complex of $\mathscr{E}$. Its cohomologies are denoted by $H_h^q(\mathscr{E})$.

Cocycles in the term $\Lambda_h^{n-1}(\mathscr{E})$ are called conservation laws of $\mathscr{E}$. A conservation law ω is trivial if $\omega = d_h\rho$. Two conservation laws are equivalent if their difference is a trivial one. Thus, $H_h^{n-1}(\mathscr{E}) = \mathrm{CL}(\mathscr{E})$, the space of equivalence classes of conservation laws.

Conservation laws: motivation

Let $\mathscr{E} = \{u_t = F[u]\}$ be an evolutionary equations in x and t. Assume that for all functions under consideration $\lim_{x\to\infty} f(x,t) = 0$. A conservation law of $\mathscr{E}$ is a 1-form $\omega = X\,dx + T\,dt$, s.t. $D_x(T) = D_t(X)$. Then we have

$$D_t\left(\int_{-\infty}^{+\infty} X\,dx\right) = \int_{-\infty}^{+\infty} D_t(X)\,dx = \int_{-\infty}^{+\infty} D_x(T)\,dx$$
$$= T(+\infty) - T(-\infty) = 0,$$

i.e., the quantity $\mathscr{X} = \int_{-\infty}^{+\infty} X\,dx$ is constant in time (on solutions) and X is a "conserved density". If ω is trivial, then $X = D_x(G)$, $T = D_t(G)$ for some potential G and $\mathscr{X} = 0$.

Conservation laws and cosymmetries

Now, the line $n-1$ in $E_1(\mathscr{E})$ may be presented as

$$\begin{array}{ccccccc} \Lambda_h^{n-1} & \xrightarrow{\quad\mathrm{g}\quad} & & E_1^{1,n-1} & \xrightarrow{d_1} & E_1^{2,n-1} & \xrightarrow{d_1} \dots \\ & \searrow{\scriptstyle\Pi} & & \nearrow{\scriptstyle d_1} & & & \\ & & \mathrm{CL}(\mathscr{E}) = E_1^{0,n-1} & & & & \end{array}$$

where Π is the natural projection. For 2-line equations this complex in exact in the term $E_1^{0,n-1}$ and hence a conservation law ω is trivial iff $\mathrm{g}(\omega) = 0$. The element $\psi_\omega = \mathrm{g}(\omega)$ is called the generating section of ω. The term $E_1^{1,n-1}$ admits a simple description: $E_1^{1,n-1} = \ker \ell_{\mathscr{E}}^*$. Its elements are called *cosymmetries*: $E_1^{1,n-1}(\mathscr{E}) = \operatorname{cosym} \mathscr{E}$. Thus $\ell_{\mathscr{E}}^*(\psi_\omega) = 0$.

Conservation laws and cosymmetries

When $\mathscr{E}$ is a system of evolution equations

$$u_t^j = F^j[u], \qquad u = (u^1, \dots, u^m), \quad j = 1, \dots, m,$$

and

$$\omega = X\, dx^1 \wedge \cdots \wedge dx^n + \sum T^i dt \wedge dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^n$$

is a conservation law, then

$$\psi_\omega = \left(\frac{\delta X}{\delta u^1}, \dots, \frac{\delta X}{\delta u^m} \right),$$

where $\delta / \delta u^j$ is the variational derivative

$$\frac{\delta X}{\delta u^j} = \sum_\sigma (-1)^{|\sigma|} D_\sigma \frac{\partial X}{\partial u_\sigma^j}.$$

Example: KdV

Solving equation $\ell^*_{\mathscr{E}}(\psi) = 0$ up to order 6, one gets

$$\psi_1 = 1, \qquad \psi_4 = u_4 + \frac{5}{3}uu_2 + \frac{5}{6}u_1^2 + \frac{5}{18}u^3,$$
$$\psi_2 = u, \qquad \psi_5 = u_6 + \frac{7}{3}uu_4 + \frac{14}{3}u_1u_3 + \frac{7}{2}u_2^2 + \frac{35}{18}u^2u_2 \ldots,$$
$$\psi_3 = u_2 + \frac{1}{2}u^2, \quad \psi_6 = tu + x.$$

The corresponding conserved densities are

$$X_1 = u, \qquad X_4 = u_2^2 - \frac{5}{3}uu_1^2 + \frac{5}{36}u^4,$$
$$X_2 = u^2, \qquad X_5 = u_3^2 - \frac{7}{3}uu_2^2 + \frac{35}{18}u^2u_1^2 - \frac{7}{108}u^5,$$
$$X_3 = u_1^2 - \frac{1}{3}u^3, \qquad X_6 = xu + \frac{1}{2}tu^2.$$

Conservation laws and cosymmetries

For general equations, the construction is more complicated. Namely, let $\omega \in \Lambda_h^{n-1}(\mathscr{E})$ be a conservation law, i.e., $d_h(\omega) = 0$. Extend ω to a form $\bar{\omega} \in J^\infty(\pi)$ in the ambient jets. Then

$$d_h(\bar{\omega}) = \Delta(F),$$

where F is such that $\mathscr{E} = \{F = 0\}$ and Δ is a $\mathscr{C}$-differential DO. Then

$$\psi_\omega = \Delta^*(1)|_{\mathscr{E}} \in \operatorname{cosym}(\mathscr{E}).$$

Cosymmetries and variational symplectic structures

Let $\psi \in \operatorname{cosym} \mathscr{E}$ be a cosymmetry and $\bar{\psi}$ be its extension on the ambient jets. Then, since $\ell^*_{\mathscr{E}}(\psi) = 0$, one has

$$\ell^*_F(\bar{\psi}) = \bar{\Delta}(F),$$

where $\bar{\Delta}$ is a $\mathscr{C}$-differential operator. Let $\Delta = \bar{\Delta}\big|_{\mathscr{E}}$. Then its restriction

$$\mathscr{S}_\psi = \ell_\psi - \Delta^* = d_1(\psi) \in E_1^{2,n-1}.$$

The operator $\mathscr{S}_\psi\colon \operatorname{sym} \mathscr{E} \to \operatorname{cosym} \mathscr{E}$ is a local *variational symplectic structure* and defines a Poisson bracket $\{\cdot, \cdot\}_\psi$ in the space of *admissible* conservation laws:

$$d_1\{\omega_1, \omega_2\}_\psi = \mathscr{S}_\psi([S_1, S_2]), \quad S_1, S_2 \in \operatorname{sym} \mathscr{E},$$

where $d_1 \omega_i = \mathscr{S}_\psi(S_i)$, $i = 1, 2$.

Differential coverings. Motivations

Initially, existence of unlimited number of KdV higher symmetries was discovered experimentally, in direct computations. Later it was showed that they may be obtained by means of the so-called Lenard recursion operator

$$\mathscr{R} = D_x^2 + \frac{2}{3}u + \frac{1}{3}u_x D_x^{-1}$$

Indeed, all the $\mathscr{R}^k(u_x)$ are well defined symmetries. But $\mathscr{R}(\varphi_6)$, $\varphi_6 = tu_3 + \frac{1}{3}(3tu + x)u_1 + \frac{2}{3}u$, contains the term $D_x^{-1}(u)$, which at first glance makes no sense in our geometric paradigm. A simple idea: introduce w, s.t. $w_x = u$. Rigorously, it is realized in the theory of differential coverings.

By the way, the hieroglyph "D_x^{-1}" also needs clarification!

Differential coverings. Linguistic game & definition

Generically, infinite prolongations $\mathscr{E}$ are infinite-dimensional. But there is another, "differential dimension" of $\mathscr{E}$: that of the Cartan distribution.

A surjection $\tau\colon \tilde{M} \to M$ of two finite-dimensional manifolds is a covering if $\dim \tau^{-1}(x) = 0$, $x \in M$. Hence, $\tau\colon \tilde{\mathscr{E}} \to \mathscr{E}$ is a *differential covering* if differential dimension of fibers is zero.

Definition: Let $\tau\colon \tilde{\mathscr{E}} \to \mathscr{E}$ be a fiber bundle, s.t. $\tau_*(\tilde{\mathscr{C}}_{\tilde{\theta}}) \subset \mathscr{C}_{\tau(\tilde{\theta})}$. It is a (differential) covering if

$$\tau_*|_{\tilde{\mathscr{C}}_{\tilde{\theta}}} : \tilde{\mathscr{C}}_{\tilde{\theta}} \simeq \mathscr{C}_{\tau(\tilde{\theta})}$$

for all $\tilde{\theta} \in \tilde{\mathscr{E}}$. Two coverings are equivalent if

$$\begin{array}{ccc} \tilde{\mathscr{E}}_1 & \xrightarrow{f} & \tilde{\mathscr{E}}_2 \\ & {\scriptstyle\tau_1}\searrow \quad \swarrow{\scriptstyle\tau_2} & \\ & \mathscr{E}, & \end{array}$$

where f is an isomorphism of equations.

Differential coverings

All objects on $\tilde{\mathscr{E}}$ are *nonlocal* w.r.t. $\mathscr{E}$. In particular, φ: $\ell_{\tilde{\mathscr{E}}}(\varphi) = 0$ and ψ: $\ell^*_{\tilde{\mathscr{E}}}(\psi) = 0$ are nonlocal symmetries and cosymmetries, resp. On the other hand, any $\mathscr{C}$-differential operator Δ on $\mathscr{E}$ admits a natural lift $\tilde{\Delta}$ to $\tilde{\mathscr{E}}$, and solutions of

$$\tilde{\ell}_{\mathscr{E}}(\varphi) = 0, \quad \tilde{\ell}^*_{\mathscr{E}}(\psi) = 0$$

are called *shadows* of symmetries and cosymmetries in τ.

Let w^α be fiber-wise coordinates (*nonlocal variables*) in some trivialization of τ. Then $\tilde{D}_i = D_i + X_i$, where

$$X_i = \sum_i X_i^\alpha \frac{\partial}{\partial w^i}$$

and τ is a covering iff

$$D_i(X_j) - D_j(X_i) + [X_i, X_j] = 0, \quad i < j.$$

Differential coverings

To any conservation law ω there corresponds a one-dimensional covering τ_ω. In the 2D case this correspondence is simple: if $\omega = X_1\,dx^1 + X_2\,dx^2$, then

$$\tilde{D}_1 = D_1 + X_1 \frac{\partial}{\partial w}, \quad \tilde{D}_2 = D_2 + X_2 \frac{\partial}{\partial w}.$$

Moreover, if $\omega_1 \simeq \omega_2$, then $\tau_{\omega_1} \simeq \tau_{\omega_2}$.

Example: Let $\omega = u\,dx + \left(\frac{1}{2}u^2 + u_2\right) dt$ be the 1st conservation law of KdV. The corresponding covering τ_ω is given by

$$\tilde{D}_x = D_x + u\frac{\partial}{\partial w}, \quad \tilde{D}_t = D_t + \left(\frac{1}{2}u^2 + u_2\right)\frac{\partial}{\partial w}$$

and $w_x = u$ in this covering. Action of $\mathscr{R}$ on φ_6 produces a τ_ω-shadow and makes sense now. Note also that $\tilde{\mathscr{E}}$ in this case is

$$w_t = \frac{1}{2}w_x^2 + w_{xxx},$$

the potential KdV.

Bäcklund transformations

In 1880, V. Bäcklund found out that the system

$$v_y - u_x = 2\lambda \sin\left(\frac{v+u}{2}\right), \; v_y + u_x + \frac{2}{\lambda} a \sin\left(\frac{v-u}{2}\right), \; 0 \neq \lambda \in \mathbb{R},$$

possesses a remarkable property: if u is a solution of the sine-Gordon equation $u_{xy} = \sin u$, then v is a solution as well and vice versa. Later, similar relations were found for other equations (Liouville, KdV, etc.) and exact solutions (multi-soliton, multi-kink) were constructed. In the language of coverings, a BT is a diagram

$$\mathcal{E}_1 \xleftarrow{\tau_2} \tilde{\mathcal{E}} \xrightarrow{\tau_1} \mathcal{E}_2$$

Example: The Cauchy–Riemann system is an auto-BT of the Laplace equation:

$$\boxed{w_{xx} + w_{yy} = 0} \xleftarrow{\substack{w_x = u \\ w_y = -v}} \boxed{u_x = v_y, \; u_y = -v_x} \xrightarrow{\substack{w_x = v \\ w_y = u}} \boxed{w_{xx} + w_{yy} = 0}$$

Tangent and cotangent coverings

Consider $\mathscr{E}$ and set $\mathbf{t}\colon \mathscr{T}(\mathscr{E}) = T\mathscr{E}/\mathscr{C} \to \mathscr{E}$. Then $\mathbf{t}$ is the tangent covering. In adapted coordinates,

$$\mathscr{T}(\mathscr{E})\colon \qquad \begin{array}{l} \ell_F(q) = 0, \\ F = 0; \end{array} \qquad \mathbf{t}\colon (u,q) \mapsto u.$$

Unfortunately, I do not know a coordinate-free definition of the dual object and define the cotangent covering as

$$\mathscr{T}^*(\mathscr{E})\colon \qquad \begin{array}{l} \ell_F^*(p) = 0, \\ F = 0; \end{array} \qquad \mathbf{t}^*\colon (u,p) \mapsto u.$$

It can be proved that $\mathscr{T}^*(\mathscr{E})$ is well defined for two-line equations.

Important remark: The variables p and q must be treated as *odd* of parity 1. Otherwise everything breaks down.

Tangent and cotangent coverings. Properties

- Sections of $\mathbf{t}$ that preserve the Cartan distribution are identified with symmetries of $\mathscr{E}$
- q-linear conservation laws of $\mathscr{T}\mathscr{E}$ are identified with cosymmetries of $\mathscr{E}$
- q-linear shadows of symmetries in $\mathbf{t}$ are local recursion operators for symmetries of $\mathscr{E}$
- q-linear shadows of cosymmetries in $\mathbf{t}$ are loc. symplectic structures of $\mathscr{E}$
- Sections of $\mathbf{t}^*$ that preserve the Cartan distribution are identified with cosymmetries of $\mathscr{E}$
- p-linear conservation laws or $\mathscr{T}^*\mathscr{E}$ are identified with symmetries of $\mathscr{E}$
- p-linear shadows of symmetries in $\mathbf{t}^*$ are local Poisson structures of $\mathscr{E}$
- p-linear shadows of cosymmetries in $\mathbf{t}^*$ are local recursion operators for cosymmetries of $\mathscr{E}$

Tangent and cotangent coverings. Properties

Note also that

- There exists a nilpotent vector field $\mathbf{X}$ on $\mathscr{T}\mathscr{E}$ induced by the Cartan differential d_v
- $\mathscr{T}^*\mathscr{E}$ is always an Euler–Lagrange equation with the Lagrangian density $\langle F, p\rangle$. Hence, $\ell_{\mathscr{T}^*\mathscr{E}} = \ell^*_{\mathscr{T}^*\mathscr{E}}$ and

$$\operatorname{sym}\mathscr{T}^*\mathscr{E} = \operatorname{cosym}\mathscr{T}^*\mathscr{E}$$

  The identity operator is a canonical symplectic structure on $\mathscr{T}^*\mathscr{E}$

Tangent and cotangent coverings. Integrable structures

Consider the following Bäcklund transformations

$$\begin{array}{ccc} & W & \\ \swarrow & & \searrow \\ \mathcal{T}\mathcal{E} & & \mathcal{T}\mathcal{E}, \end{array} \quad \begin{array}{ccc} & W & \\ \swarrow & & \searrow \\ \mathcal{T}\mathcal{E} & & \mathcal{T}^*\mathcal{E}, \end{array} \quad \begin{array}{ccc} & W & \\ \swarrow & & \searrow \\ \mathcal{T}^*\mathcal{E} & & \mathcal{T}^*\mathcal{E} \end{array}$$

Then, due to the properties of the tangent and cotangent coverings,

- The left one is a nonlocal recursion operator for symmetries of $\mathcal{E}$
- The middle one, with additional properties, is a nonlocal symplectic ($\longrightarrow$) or nonlocal Poisson ($\longleftarrow$) operator for $\mathcal{E}$
- The right one is a nonlocal recursion operator for cosymmetries of $\mathcal{E}$
- If one of arrows is the identity, the structure becomes local

Example. KdV

One has

$$\mathcal{T}\mathcal{E}: \quad \begin{aligned} u_t &= uu_x + u_{xxx} \\ q_t &= u_x q + uq_x + q_{xxx} \end{aligned} \qquad \mathcal{T}^*\mathcal{E}: \quad \begin{aligned} u_t &= uu_x + u_{xxx} \\ p_t &= up_x + p_{xxx} \end{aligned}$$

RO for symmetries: The form $\omega q\,dx + (q_{xx} + uq)\,dt$ is a conservation law of $\mathcal{T}\mathcal{E}$. Consider the corresponding covering $\tau_1 = \tau_\omega$ with the nonlocal variable w_1:

$$w_{1,x} = q, \qquad w_{1,t} = q_{xx} + uq.$$

There are two independent linear shadows of symmetries in this covering:

$$R_0 = q, \qquad R_1 = \frac{2}{3}uq + q_{xx} + \frac{1}{3}u_x w_1$$

The 1st one is local and corresponds to the identical RO; the second one defines the Lenard RO

$$\mathcal{R} = D_x^2 + \frac{2}{3}u + \frac{1}{3}u_x D_x^{-1}$$

Example

Symplectic structures: There exists another conservation law on $\mathcal{T}\mathcal{E}$:

$$\omega = up\,dx + ((u^2 + u_{xx})p - u_x p_x + u p_{xx})\,dt$$

with the corresponding covering

$$w_{2,x} = up, \qquad w_{2,t} = (u^2 + u_{xx})p - u_x p_x + u p_{xx}$$

and the two-dimensional covering with the nonlocal variables w_1, w_2. There are two linear shadow of cosymmetries in this covering

$$S_1 = w_1, \qquad S_2 = p_x + \frac{1}{3} u w_1 + \frac{1}{6} w_2$$

to which the nonlocal symplectic structures

$$\mathcal{S}_1 = D_x^{-1}, \qquad \mathcal{S}_2 = D_x + \frac{1}{3} u D_x^{-1} + \frac{1}{6} D_x^{-1} \circ u$$

correspond.

Example

There are two independent p-linear shadows of symmetries in $\mathscr{T}^*\mathscr{E}$:

$$P_1 = p_x, \qquad P_2 = p_{xxx} + \frac{2}{3}up_x - \frac{1}{3}u_x p.$$

Two *local* Poisson structures correspond to them

$$\mathscr{P}_1 = D_x, \qquad \mathscr{P}_2 = D_x^3 + \frac{2}{3}uD_x - \frac{1}{3}u_x.$$

They define two compatible Poisson bracket in the space of conservation laws

$$d_1\{\omega_1, \omega_2\}_i = [\mathscr{P}_i d_1\omega_1, \mathscr{P}_i d_1\omega_2], \qquad i = 1, 2,$$

and all the conservion laws are in involution w.r.t. both (here d_1 is the differential in E_1 of Vinogradov's spectral sequence).

Example

Finally,

$$\omega = u_x p\,dx + ((uu_x + u_{xxx})p - u_x p_x + up_{xx})\,dt$$

is a conservation law of $\mathcal{T}^*\mathcal{E}$ with the corresponding covering

$$w_x = u_x p, \qquad w_t = (uu_x + u_{xxx})p - u_x p_x + up_{xx}.$$

There are two linear shadows of cosymmetries in this covering

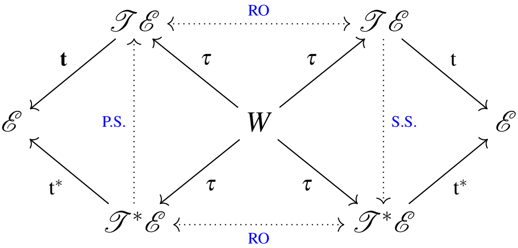
$$R_0 = p, \qquad R_1 = p_{xx} + \frac{2}{3}up - \frac{1}{3}w.$$

The corresponding ROs are

$$\mathscr{R}_0 = \mathrm{id}, \qquad R_1 = D_x^2 + \frac{2}{3}u - \frac{1}{3}D_x^{-1}u_x.$$

Finale

So, essentially the entire theory of integrable systems is encoded in the diagram of coverings



!!! Actually, there is no difference between symplectic and Poisson structures in nonlocal theory

THANK YOU FOR YOUR ATTENTION

&

A HAPPY BIRTHDAY TO A.T.F.