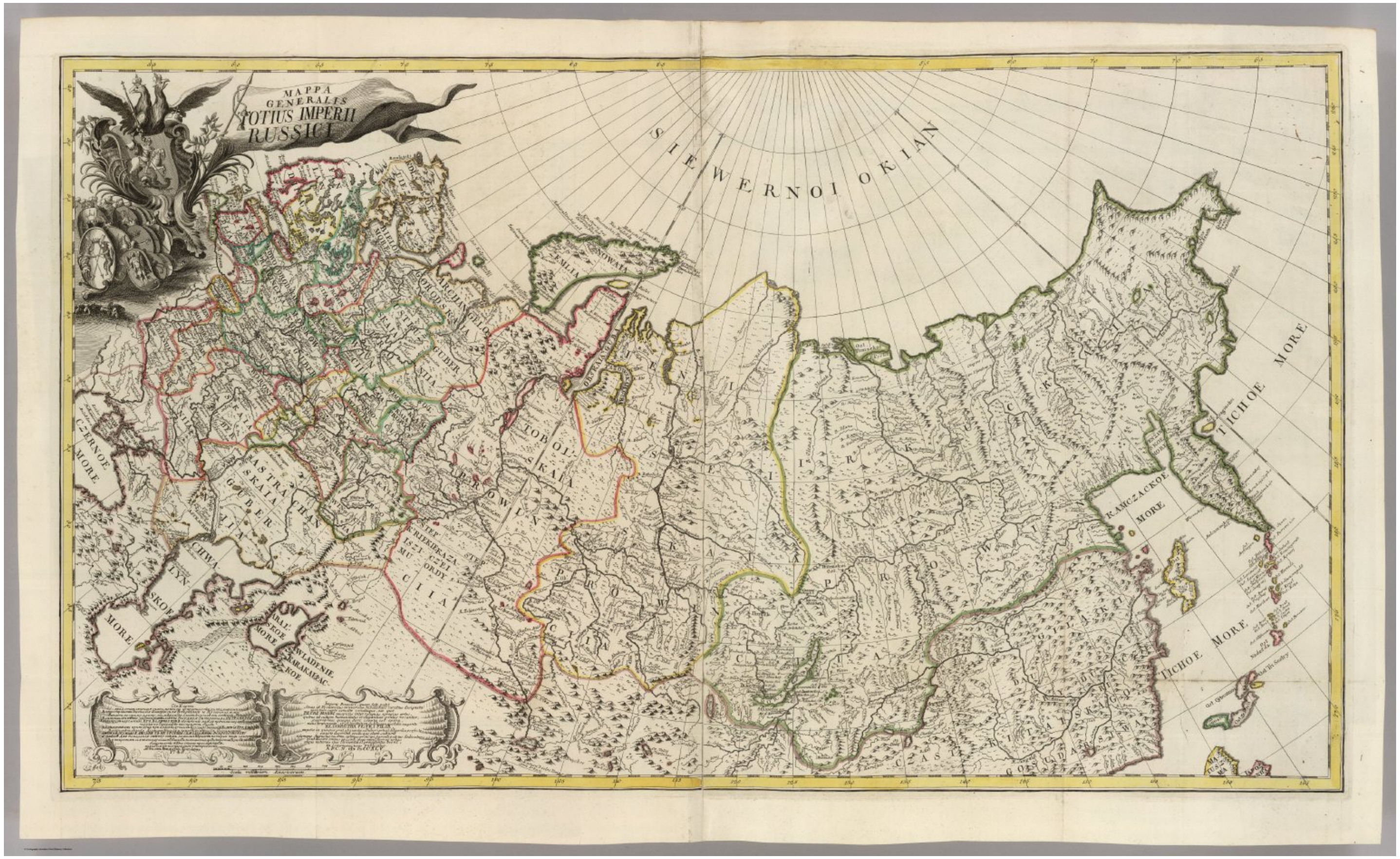
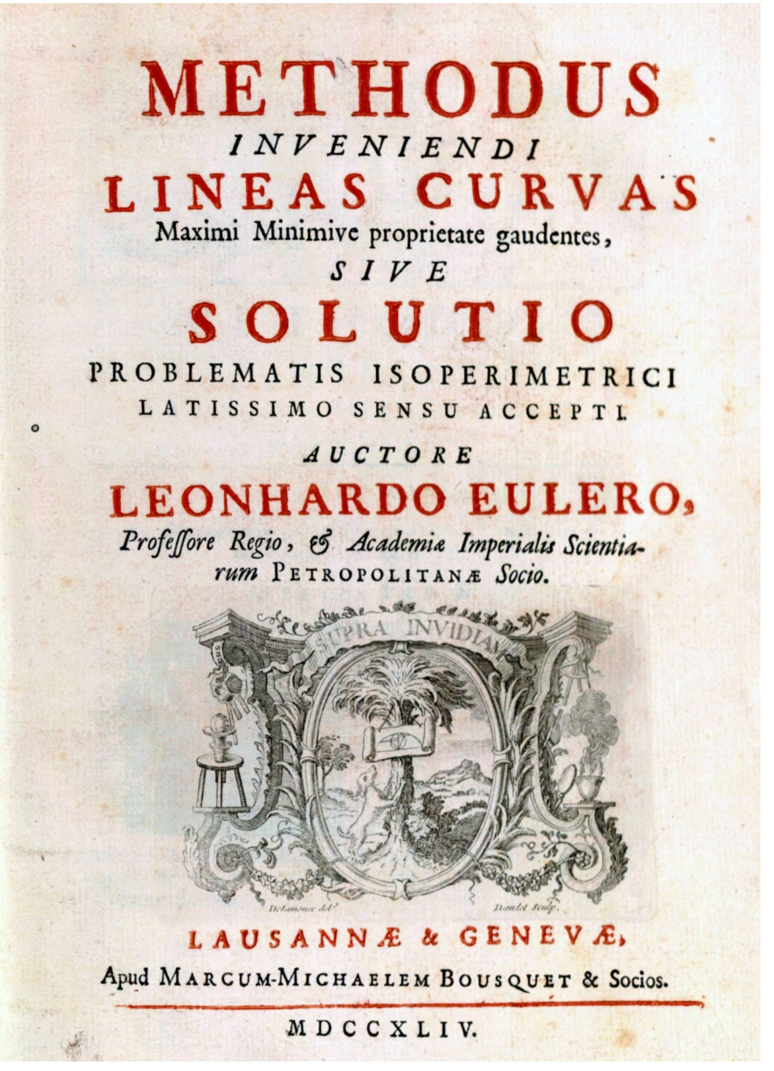
V.Yu.Protasov (L'Aquila, Moscow State University)

Geodesics on polyhedra: new results



The first map of Russian Empire (1741)



METHODUS
INVENIENDI
LINEAS CURVAS
Maximi Minimive proprietate gaudentes,
SIVE
SOLUTIO
PROBLEMATIS ISOPERIMETRICI
LATISSIMO SENSU ACCEPTI.
AUCTORE
LEONHARDO EULERO,
Professore Regio, & Academiæ Imperialis Scientiarum PETROPOLITANÆ *Socio.*

LAUSANNÆ & GENEVÆ,
Apud MARCUM-MICHAELEM BOUSQUET & Socios.
MDCCXLIV.

L.Euler, ``Methods of finding curves possessing properties of either maxima or minima'' (1744)

Jules Henri Poincaré (1854 - 1912)

Conjecture (1905). An arbitrary smooth convex surface possesses at least three closed simple geodesics.

Let us have a convex surface G (the boundary of a convex body) in $\mathbb{R}^3$.

A curve γ on G is a geodesic if it is locally shortest on the surface at every point.

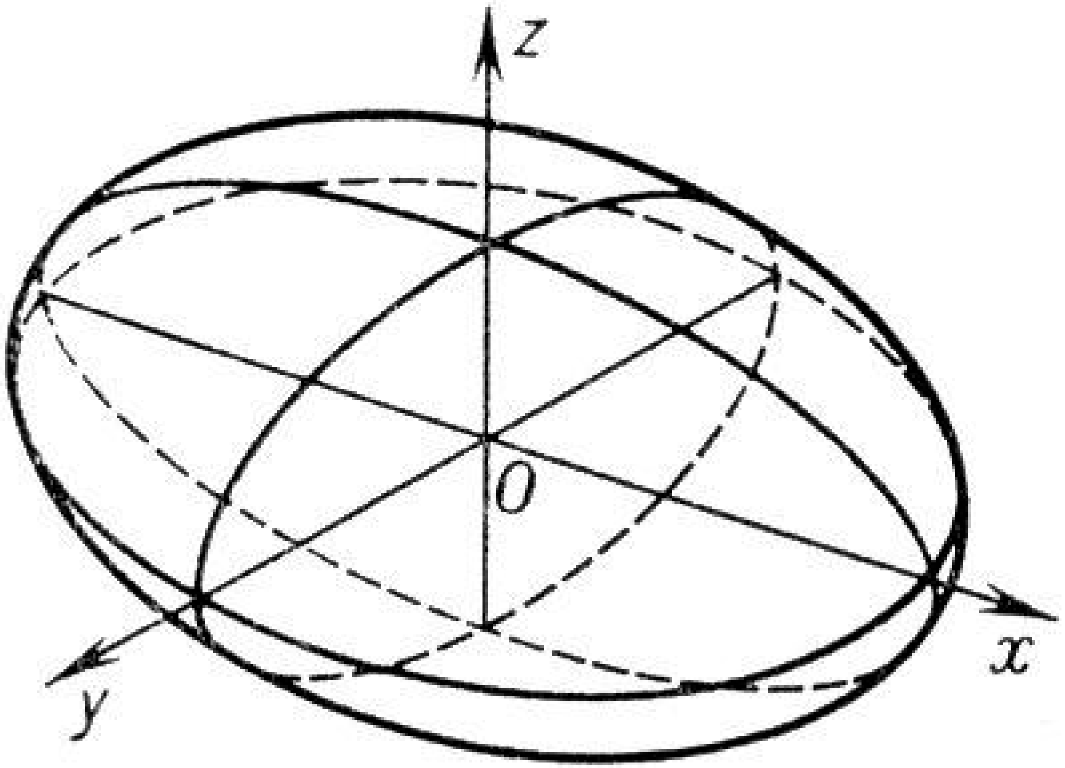
We consider only closed simple (non-self-intersecting) geodesics.

A.Poincare (1905) conjectured the existence of at least three geodesics on any smooth convex surface.

L.Lyusternik and L.Shnirelman (1930) proved the conjecture.

J.Franks (1992), V.Bangert (1993) proved the existence of infinitely many geodesics on any smooth convex surface.

What about an ellipsoid ?

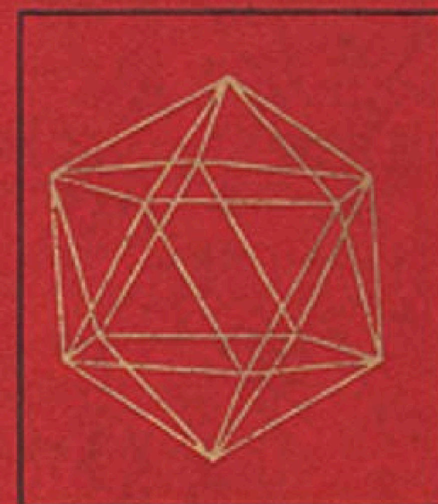


Three simple closed geodesics on an ellipsoid

Do there exist more?

Д. ГИЛЬБЕРТ
С. КОН-ФОССЕН

НАГЛЯДНАЯ ГЕОМЕТРИЯ



Не удается отобразить рисунок.

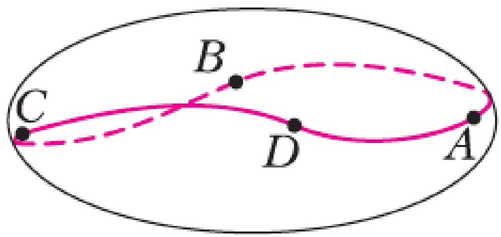
бой поверхности вращения. У них меридианы представляют всегда геодезические линии, так как плоскости меридианов проходят через ось и потому пересекают поверхность под прямым углом (выше мы уже доказали, что меридианы служат также линиями кривизны поверхности). Отсюда следует, что все замкнутые поверхности вращения обладают семейством замкнутых геодезических с одним параметром. На других поверхностях существуют только отдельные линии такого рода. Так, можно показать, что на трехосном эллипсоиде единственными замкнутыми геодезическими линиями без самопересечений являются те три эллипса, которые получаются в пересечении этой поверхности с тремя плоскостями симметрии.

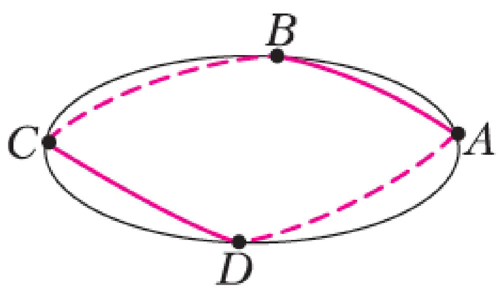
Обратно, имеет место теорема, уже давно предполагавшаяся, но лишь в 1930 г. доказанная Люстерником и Шнирельманом [1]), что на всякой выпуклой замкнутой поверхности проходят по крайней мере три замкнутые геодезические линии без самопересечений.

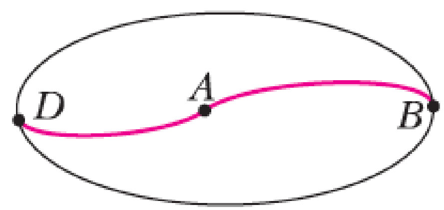
Геодезические линии имеют большое значение для физики. Материальная точка, свободная от действия сил, но вынужденная оставаться на определенной поверхности, всегда движется по геодезической линии поверхности. Каждое из приведенных нами определений геодезической линии дает основание для законов механики материальной точки; так, определение геодезической линии как кратчайшей соответствует принципу Якоби в механике; определение геодезической как прямейшей проявляется в принципе наименьшего принуждения Гаусса — Герца; соприкасающейся плоскости геодезической линии на-

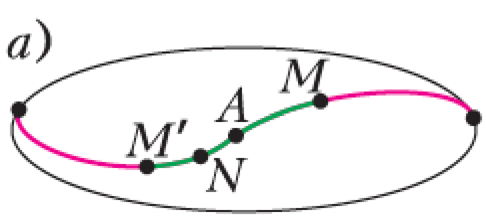
The fourth geodesics on an ellipse

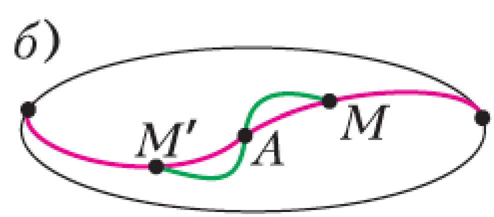
W.Klingenberg, ”Riemannian geometry” (1982)











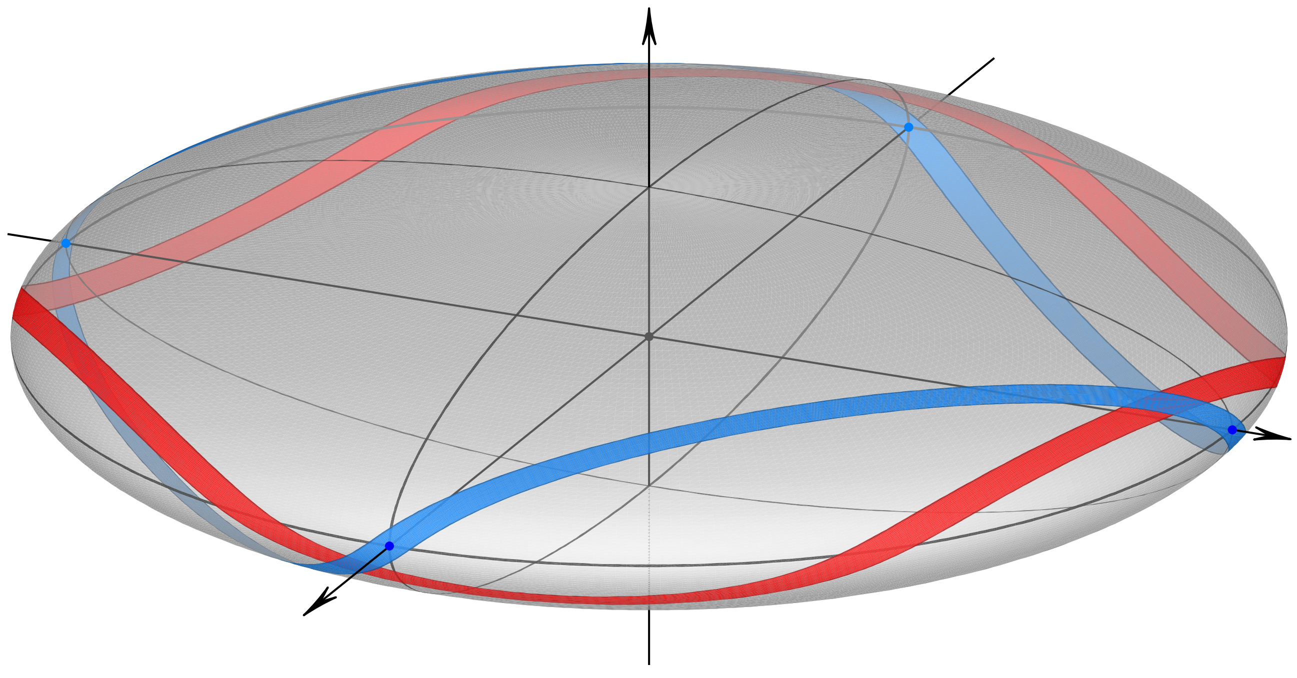
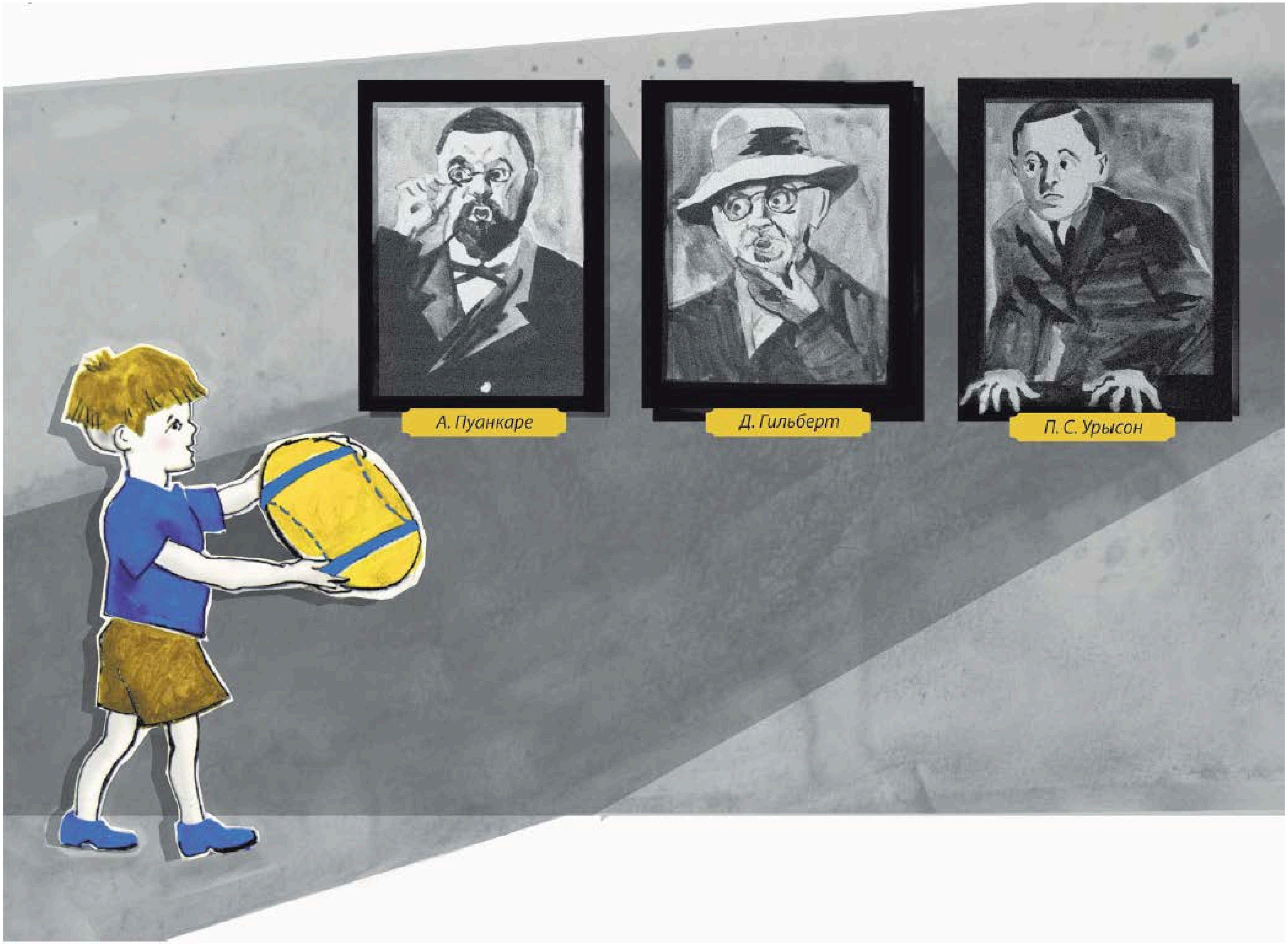


Illustration: M.Panov, N.Andreev

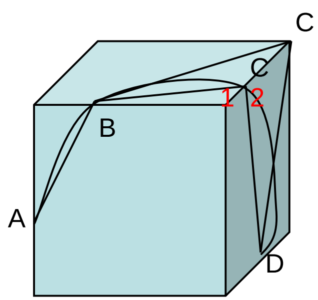

А. Пуанкаре
Д. Гильберт
П. С. Урысон

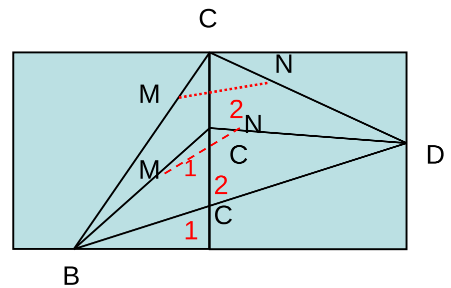
Geodesics on non-smooth convex surfaces. A.D.Alexandrov (1948), A.V.Pogorelov (1949)

Geodesics on flat surfaces and on polyhedra

K.Post, G.Galperin, A.Zorich, D.Fuchs, E. Fuchs, V.Zalgaller,

Geodesics on the surface of a polyhedron



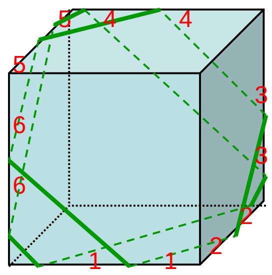


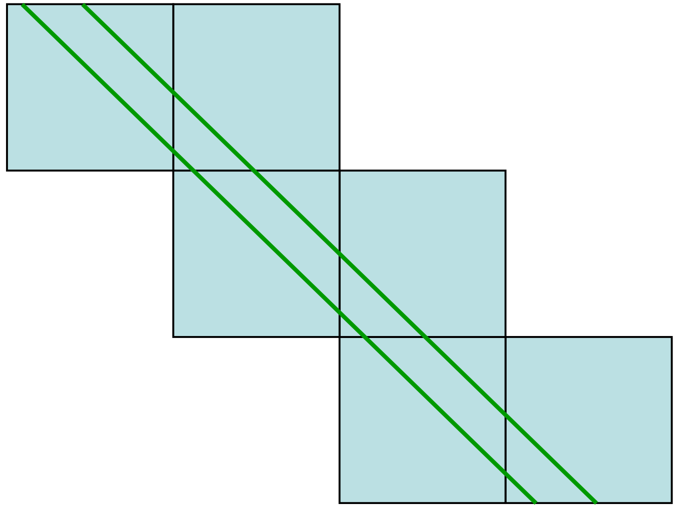
$MN < MC + CN$

angle 1 = angle 2

1. Geodesic is a broken line with nodes on the edges.
2. Geodesic does not contain vertices of the polyhedron.
3. It forms equal angles with every edge.

A closed simple curve is a geodesic iff it is a broken line with nodes on the edges that does not pass through the vertices and form equal angles with the edges.





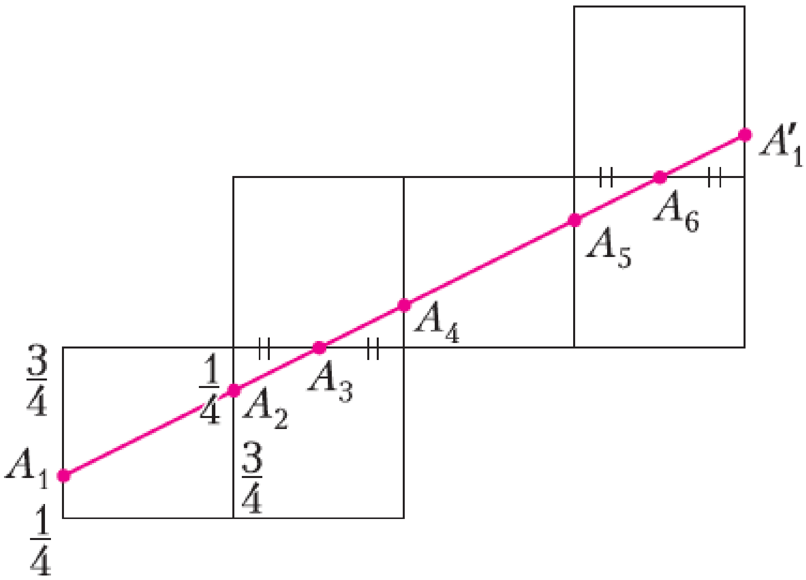
Geodesic becomes a straight line on the diagram

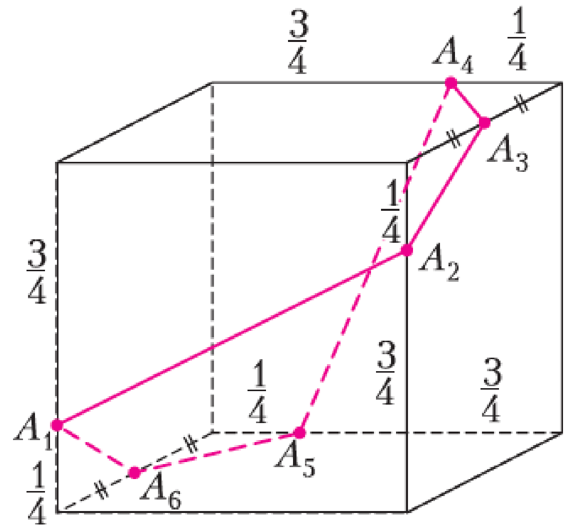
Definition. Two geodesics are isomorphic iff they intersect the same sequence of edges.

All isomorphic geodesics have parallel segments and the same length.

We identify all isomorphic geodesics

Example. A **cube** has three different geodesics.
Two of them are flat, the third one is not.



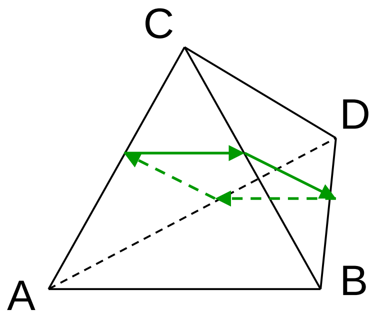


Example. An **octahedron** has two different geodesics.

Geodesics on regular polyhedra

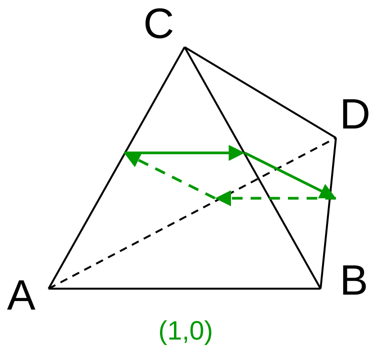
Fuchs D. Fuchs E. Moscow Math. J. (2007), 265-179
Fuchs D., Geometria Dedicata, 170 (2014), 319-333;
Fuchs D., Arnold Math. J., 2 (2016), 201-211;
Athreya J. S., Aucilino D., Amer. Math. Monthly, 126 (2019), 161-162;

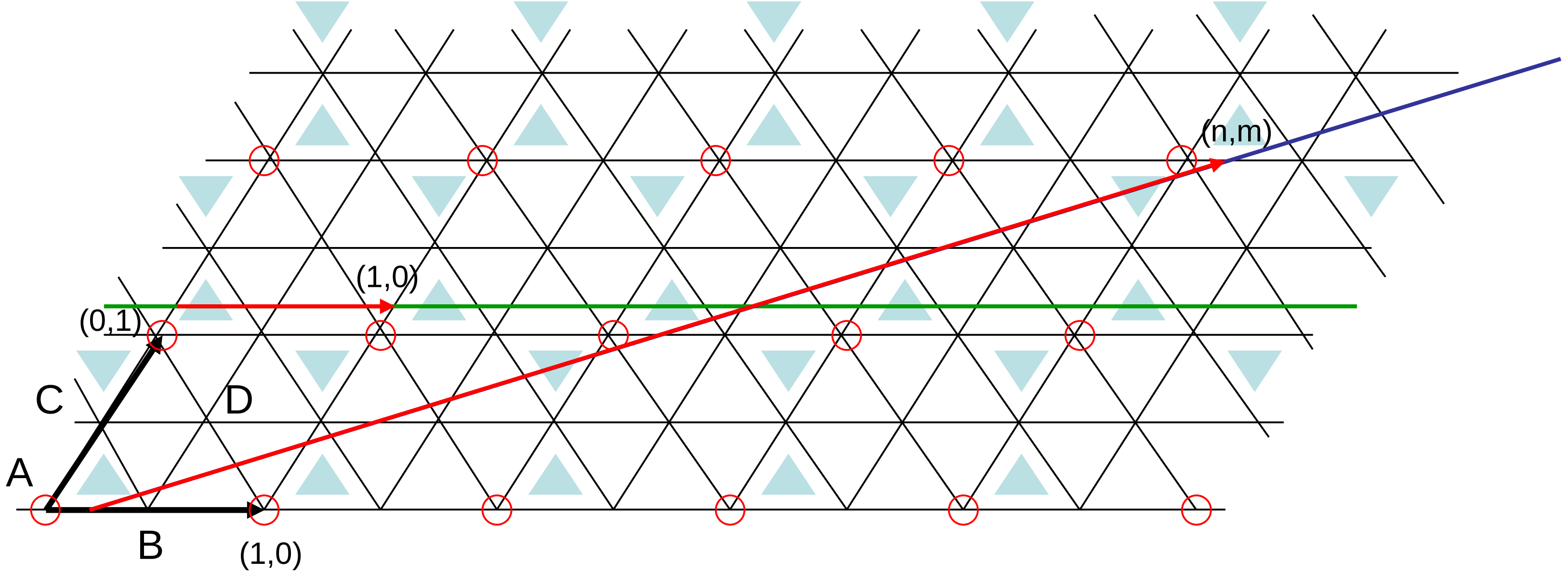
Self-intersecting geodesics have also been studied in those works.



Geodesics on a regular simplex

Geodesics on a simplex



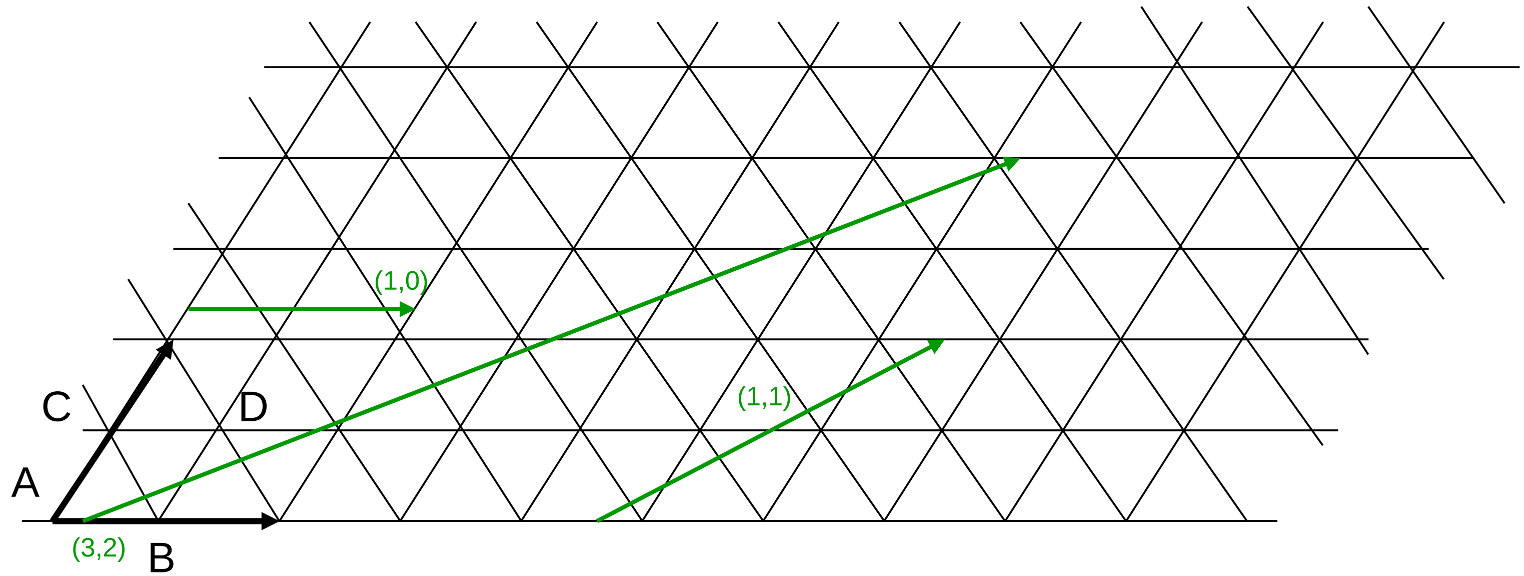


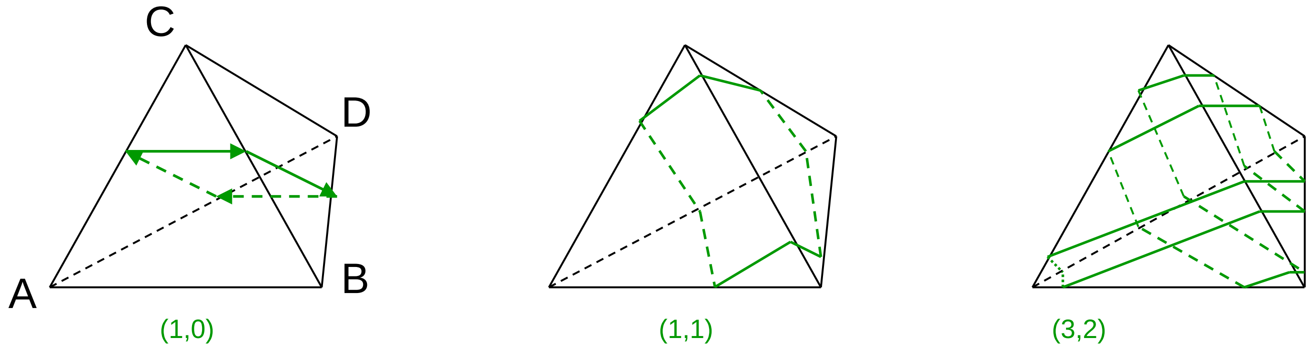
The triangular lattice – a complete net of a regular tetrahedron

Geodesic is a straight line on the net with an integer direction vector (n,m)

The (n,m)-geodesic, where either (n,m)=(1,0), or n and m are co-prime.

It has 4(n+m) vertices.





The same results hold for all isohedral simplices

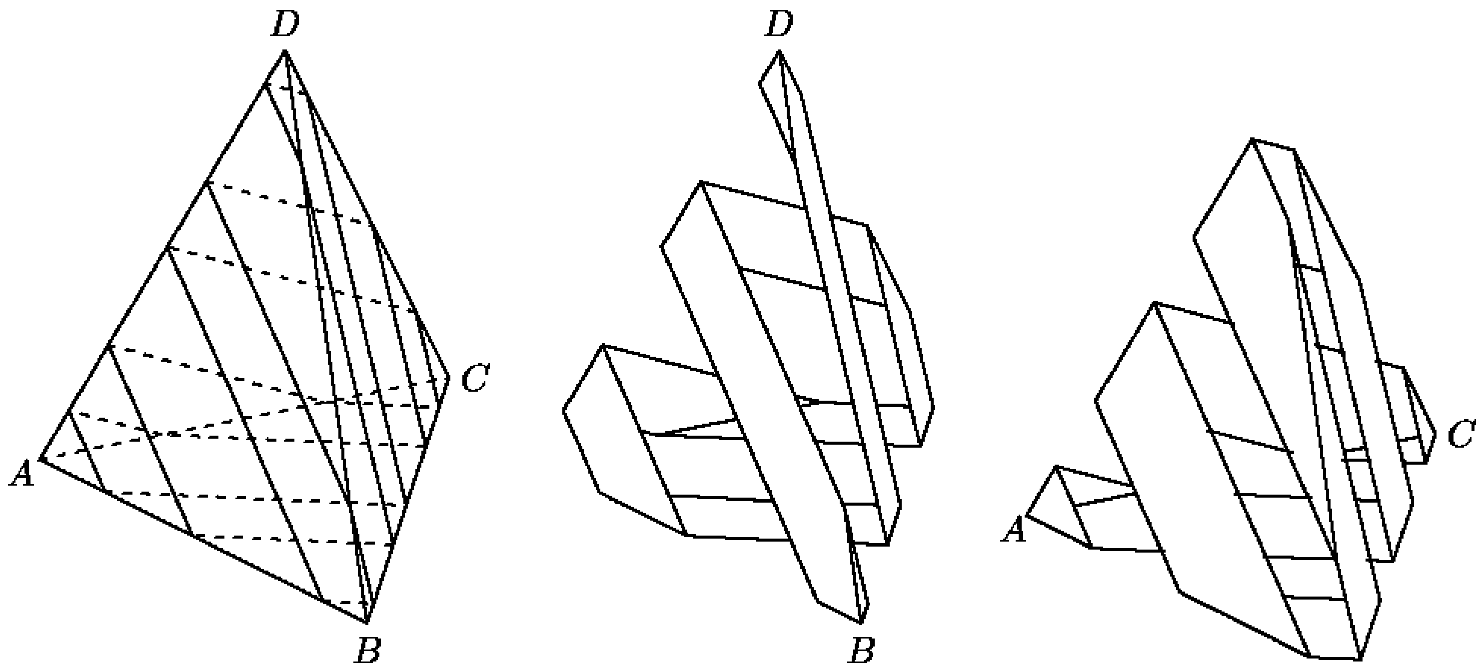


Illustration: D.Fuchs, E.Fuchs

Geodesics on an arbitrary simplex

Definition. Generalzied geodesic is a closed non-self-intersecting broken line on the surface of a simplex such that

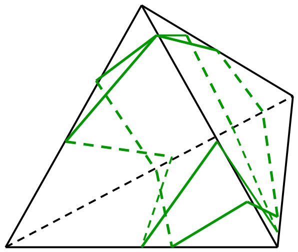
a) it has more than 3 sides;

b) it has all nodes on the edges and does not pass through the vertices of the simplex;

c) adjacent sides lie on different faces.

Theorem 1. Any generalized geodesic is isomorphic to some geodesic on the isohedral simplex.

Corollary 1. Any generalized geodesic on the isohedral simplex can be homotopically transformed to a real geodesic. This homotopy is without self-intersecting or passing through the vertices, and with a monotone decrease of the length.



Composed geodesics

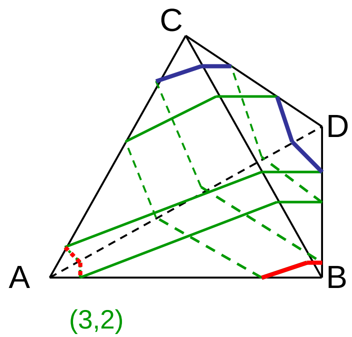
Definition. A composed geodesic is a disjoint union of finitely many simple closed geodesics. A composed generalized geodesics is defined in the same way.

Theorem 2. Every composed geodesic on an arbitrary simplex is realized on the isohedral simplex. Each of them is of type (n,m) for some n,m (or $n = 1$, $m=0$), not necessarily co-prime. If g.c.d. (m,n) = d, then the (n,m) – geodesics is composed of d parallel geodesics of type (m/d, n/d).

Corollary 2. Any composed generalized geodesic on the isohedral simplex can be homotopically transformed to a real composed geodesic. This homotopy is without self-intersecting or passing through the vertices, and with a monotone decrease of the length.

The structure of geodesics on a simplex

Definition. A geodesic *catches* a vertex A of a simplex if it successively intersects all edges going from A at the nodes that are closest to A.



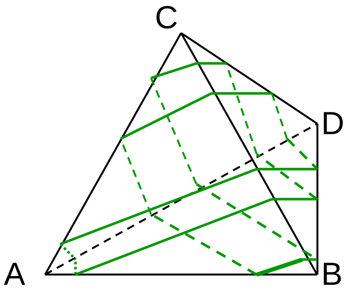
Theorem 4. Any geodesic, except for that of (1,0)-type, catches every vertex of the simplex. It splits the four vertices of the simplex into two pairs. The geodesic catches the first vertex, then form a strip on the surface, then catches the second one.

A necessary condition for the existence of a geodesic:

Theorem 5. If a simplex possesses a geodesic, then the sum of planar angles at two of its vertices is 2π.

This also follows from the results of K.Post (1970) and G.Galperin (1991, 2003).
They used the discrete version of the Gauss–Bonnet theorem.

Example. A regular triangular pyramid that is not a regular simplex has no geodesics.



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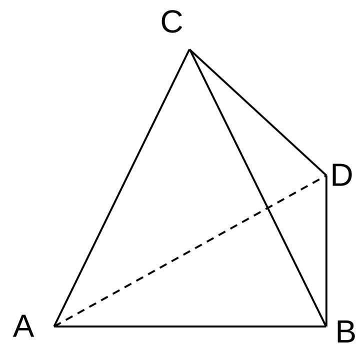
Example. A regular triangular pyramid that is not a regular simplex has no geodesics.



Geodesics on an arbitrary simplex

Theorem 6. A non-isohedral simplex can have only finitely many geodesics

Corollary 3. If a simplex possesses arbitrarily long geodesics, then it is isohedral.

Example 1. The length of any geodesic on a simplex does not exceed $\dfrac{d}{2\sin\dfrac{\delta}{2}}$,

where d is the diameter of a simplex, and δ is the maximal deviation of the sum of three planar angles at a vertex from π. This estimate is sharp.

Example 2. The total number of different geodesics on a simplex does not exceed $4\,\psi\left(2 + 3d/(4h\sin\delta/2)\right)$, where h is the shortest apotheme of the simplex, $\psi(x)$ is the number of pairs of natural co-prime numbers (n, m) such that $n+m \le x$. It is known that $\psi(x) \le \frac{3x^2}{2\pi^2} + O(x\ln x), \quad x \to \infty$.

Geodesics on an arbitrary polyhedron

Theorem 7. If a polyhedron has infinitely many non-isomorphic geodesics, then it is an isohedral simplex.

Equivalently: If a polyhedron has arbitrarily long geodesics, then it is an isohedral simplex. For all other polyhedra the lengths of geodesics are bounded above.

Problem 1. Is there a convex surface, different from an isohedral simplex that possesses arbitrarily long geodesics ?

From Theorem 5 it follows that for the set of all polyhedra the answer is negative.

A.Petrunin (2010) proved the negative answer for smooth surfaces with Strictly positive positive (separated from zero) Gauss curvature.

A.Akopyan and A.Petrunin (2018) extended this result to arbitrary convex surfaces

Geodesics on a simplex in the Lobachevsky space

Theorem 8 (A.Borisenko, D.Sukharebskata, 2019) For every co-prime (m,n) there is a unique (up to the symmetry) geodesic on a regular simplex.

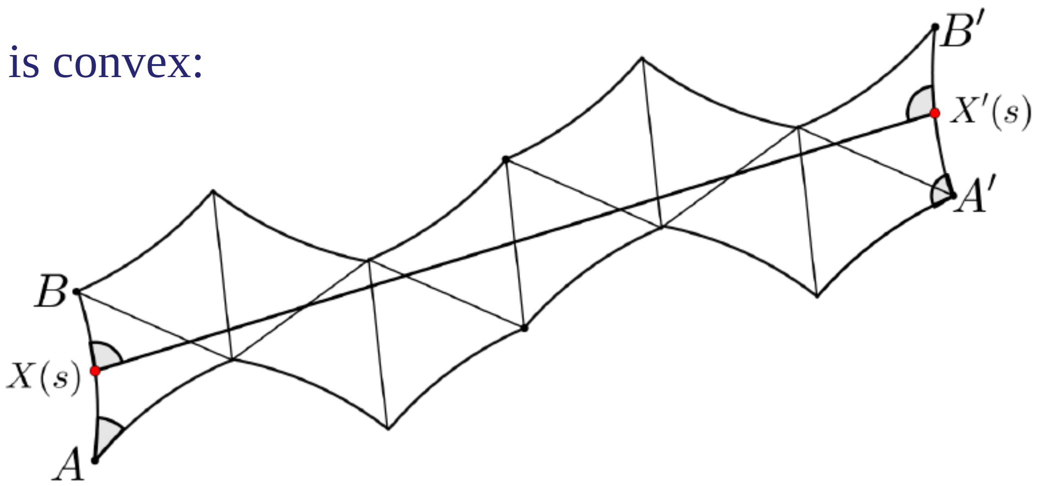
Remark. In the Euclidean space, there are infinitely many equivalent ones.

Problem 1. What about other isohedral simplices?

Theorem 9 (A.Borisenko, D.Sukharebskaya, 2020). If all flat angles of a simplex in the Lobachevsky space are smaller than 45^o, then it has a geodesic of type (m,n) for every co-prime m,n.

Remark. No equality constraints for the simplex as in the Euclidean case! The set of simplices that have geodesics is not any more a null set.

Proof. This figure is convex:



Problem 2. Thus, in the Lobachevsky space, not only isohedral simplices may possess arbitrarily long geodesics. What about other polyhedral? Convex surfaces?

Problem 3. What are sufficient conditions for the existence of at least one geodesic on a given polyhedron ?

Problem 4. To estimate the length and the total number of geodesics for an arbitrary polyhedron.

Problem 5. Is it true that for every k there exists a simplex that possesses precisely k geodesics?

THANK YOU !