QUANTISATION OF FREE ASSOCIATIVE DYNAMICAL SYSTEMS.

BI-QUANTISATION OF STATIONARY KDV HIERARCHY AND NOVIKOV'S EQUATIONS.

NON-DEFORMATION QUANTISATION OF THE VOLTERRA HIERARCHY.

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OUTLINE

1. CLASSICAL, QUANTUM AND FREE ASSOCIATIVE MECHANICS
   - Classical, Quantum Mechanics
   - Free Associative Mechanics
   - Problem of quantisation, quantisation ideals
2. QUANTISATION OF NON-ABELIAN HOMOGENEOUS QUADRATIC SYSTEMS
   - Quantisation of quadratic systems with a cubic symmetry
   - Quantisation of quadratic systems with a quartic symmetry
3. QUANTISATION OF NOVIKOV'S AND STATIONARY KDV SYSTEMS
   - Quantisation of Novikov's systems
   - Bi-quantisation of stationary KdV
   - Bi-quantisation of the $N = 1$ Novikov equation
4. QUANTISATION OF VOLTERRA AND BOGOYAVLENSKY SYSTEMS
   - Quantisation of the Volterra systems
   - Periodic Volterra chains
   - Quantisation of the periodic Volterra chains
   - Quantisation (b) of odd degree periodic Volterra symmetries
   - Quantisation of the Bogoyavlensky family of systems

Classical Mechanics (Newton, Hamilton): $\mathfrak{A}_0 = \mathbb{C}[p, q]$

$$\partial_t q = p, \;\; \partial_t p = F(q), \quad H = \frac{p^2}{2} + U(q), \;\; F(q) = -U'(q)$$

$$\partial_t q = \{q, H\}, \; \partial_t p = \{p, H\}, \;\; \{q, p\} = 1, \;\; \partial_t a = \{a, H\}, \;\; a \in \mathfrak{A}_0.$$

Quantum Mechanics (Heisenberg, Dirac 1925): $\mathfrak{A}_\hbar = \mathbb{C}\langle \hat{p}, \hat{q} \rangle \big/ \langle \hat{p}\hat{q} - \hat{q}\hat{p} + i\hbar \rangle$

$$\partial_t \hat{q} = \hat{p}, \;\; \partial_t \hat{p} = F(\hat{q}), \quad H = \frac{\hat{p}^2}{2} + U(\hat{q}), \;\; F(\hat{q}) = -U'(\hat{q})$$

$$\partial_t \hat{q} = \frac{i}{\hbar}[H, \hat{q}], \; \partial_t \hat{p} = \frac{i}{\hbar}[H, \hat{p}], \quad [\hat{q}, \hat{p}] = i\hbar, \; \partial_t a = \frac{i}{\hbar}[H, a], \;\; a \in \mathfrak{A}_\hbar.$$

The Fundamental Equations of Quantum Mechanics

Author(s): P. A. M. Dirac

Source: *Proceedings of the Royal Society of London. Series A, Containing Papers of a Mathematical and Physical Character*, Dec. 1, 1925, Vol. 109, No. 752 (Dec. 1, 1925), pp. 642-653

Free Associative Mechanics:
Free associative algebra $\mathfrak{A} = \mathbb{C}\langle p, q\rangle$ with a derivation

$$\partial_t : \mathfrak{A} \mapsto \mathfrak{A}, \qquad \partial_t(a\,b) = \partial_t(a)\,b + a\,\partial_t(b), \;\; \forall\, a, b \in \mathfrak{A}.$$

Nonabelian Newton's equations:

$$\partial_t q = p, \qquad \partial_t p = F(q) \qquad p, q \in \mathfrak{A}.$$

$H_0 = [q, p] \in \mathfrak{A}$ is a constant of motion

$$\partial_t([q, p]) = p^2 + qF(q) - F(q)q - p^2 = 0$$

of the nonabelian Newton's equations, but usual expression for the first integal of energy $H = \frac{1}{2}p^2 + U(q),\; F(q) = -U'(q)$ is not a constant of motion (if $F''(q) \neq 0$).

Example: Let $\partial_t q = p,\ \ \partial_t p = F(q) = 3q^2,\ \ U(q) = -q^3$ (*).

- $H = \frac{1}{2}p^2 - q^3$ is **not** a constant of motion

$$\partial_t(H) = \tfrac{1}{2}(\dot p p + p\dot p) - (\dot q q^2 + q\dot q q + q^2\dot q)$$

$$= \tfrac{1}{2}(pq^2 - 2qpq + q^2 p) = \tfrac{1}{2}[pq, q] + \tfrac{1}{2}[q, qp] \neq 0, \quad \partial_t(H) \in \mathrm{Span}_{\mathbb{C}}[\mathfrak{A}, \mathfrak{A}].$$

There are infinitely many algebraically independent "first integrals" in $\mathrm{Span}_{\mathbb{C}}[\mathfrak{A}, \mathfrak{A}]$.

- Nonabelian Newton's equation (*) has infinitely many higher symmetries. The next symmetry is:

$$\partial_{t_5} q = q^2 p - 2qpq + pq^2;$$
$$\partial_{t_5} p = -qp^2 + 2pqp - p^2 q;$$

$$\partial_{t_7} q = 2pq^3 - p^3 + 2q^3 p - q^2 pq - qpq^2;$$
$$\partial_{t_7} p = -2q^2 p^2 + qpqp - 2qp^2 q + pq^2 p + pqpq - 2p^2 q^2 + 6q^5;$$

In the commutative case H is a first integral and $\partial_{t_7} = 2H\partial_t$.

In the quantum case H is a constant of motion $(H)_t = 0$, since

$$-2\hat q\hat p\hat q = -2\hat p\hat q^2 - 2i\hbar\hat q, \qquad \hat q^2\hat p = \hat p\hat q^2 + 2i\hbar\hat q.$$

Algebra $\mathfrak{A}$, as a $\mathbb{C}$–linear space has an additive basis of monomials

$$Mon(\mathfrak{A}) = \{p^{i_1} q^{j_1} p^{i_2} q^{j_2} \cdots p^{i_m} q^{j_m} \mid i_k, j_k \in \mathbb{Z}_{\geq 0}\},$$

and the number of monomials of a fixed degree n is growing exponentially, as 2^n.

In contrast, algebras $\mathfrak{A}_0 = \mathfrak{A} / \langle qp - pq \rangle$ and $\mathfrak{A}_\hbar = \mathfrak{A} / \langle pq - qp + i\hbar \rangle$ have additive bases of normally ordered monomials (standard)

$$Mon(\mathfrak{A}_0) = Mon(\mathfrak{A}_\hbar) = \{p^i q^j \mid i, j \in \mathbb{Z}_{\geq 0}\}$$

respectively, and the # of monomials of degree n is growing as $n + 1$.

Any element of the quotient algebra $\mathfrak{A}_0$ or $\mathfrak{A}_\hbar$ can be uniquely represented by a polynomial with normally ordered monomials.

Problem of quantisation, quantisation ideals

Fact: Any associative $\mathbb{C}$– algebra can be represented (is isomorphic to) a quotient of a free algebra $\mathfrak{A}$ over a two sided ideal $\mathfrak{J}$.

In my opinion, the problem of *quantisation* of a dynamical system $\partial_t : \mathfrak{A} \mapsto \mathfrak{A}$ can be formulated as following:

Find such ideals $\mathfrak{J} \subset \mathfrak{A}$ that

A. $\partial_t(\mathfrak{J}) \subseteq \mathfrak{J} \quad \Leftrightarrow$ the evolutionary derivation ∂_t induces a derivation of the quotient algebra $\mathfrak{A} \diagup \mathfrak{J}$.

B. The quotient algebra $\mathfrak{A} \diagup \mathfrak{J}$ has an additive basis of normally ordered monomials. In other words, we know how to change the order of any two variables.

Ideals $\mathfrak{J}$ satisfying conditions A, B are called *quantisation ideals* and the corresponding quotient algebras $\mathfrak{A} \diagup \mathfrak{J}$ *quantised algebras*.

If we start from a classical (commutative) dynamical system, then in order to apply our method we need to lift it to a free associative algebra. This step is delicate.

Let $\mathfrak{A} = \mathbb{C}\langle x_1, \dots, x_n\rangle$ be a free associative $\mathbb{C}$ algebra. It has the additive basis of monomials

$$Mon(\mathfrak{A}) = \left\{ x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \dots x_{i_m}^{\alpha_m} \,|\, 1 \le i_1, i_2, \dots, i_m \le n,\ \alpha_k \ge 0 \right\}$$

Let $\mathfrak{J} = \langle f_{i,j} \,|\, 1 \le i < j \le n\rangle$ where

$$f_{i,j} = x_i x_j - \omega_{i,j} x_j x_i + \xi_{i,j}, \quad \xi_{i,j} \in \mathfrak{A},\ Lm(\xi_{i,j}) < x_j x_i < x_i x_j.$$

Then in the quotient $\mathfrak{A}/\mathfrak{J}$ there is a monomial basis of normally ordered monomials

$$Mon(\mathfrak{A}/\mathfrak{J}) = \left\{ x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \dots x_{i_m}^{\alpha_m} \,|\, 1 \le i_1 < i_2 < \dots < i_m \le n,\ \alpha_k \ge 0 \right\}$$

Algebra $\mathfrak{A}/\mathfrak{J}$ has Poincarée–Birkhoff–Witt basis of normally ordered monomials $\Rightarrow \mathfrak{A}/\mathfrak{J}$ satisfies (B) condition.

PROBLEM OF QUANTISATION, QUANTISATION IDEALS

In our example of the Newton equation on $\mathfrak{A}$

$$\partial_t q = p, \qquad \partial_t p = 3q^2 \qquad p, q \in \mathfrak{A}. \tag{1}$$

a natural candidate for $\mathfrak{J}$ which implies B is

$$\mathfrak{J} = \langle J := pq - \omega qp + \alpha p + \sum_{n=1}^{M} \beta_n q^n + \gamma \rangle, \quad \omega, \alpha, \beta, \gamma \in \mathbb{C}.$$

Then $\mathfrak{J} = \{\sum a_i J\, b_i \mid a_i, b_i \in \mathfrak{A}\}$. The condition (A): $\partial_t(\mathfrak{J}) \subseteq \mathfrak{J} \;\Leftrightarrow \partial_t(J) \in \; \mathfrak{J}$ implies that

$$\partial_t(J) = \dot{p}q + p\dot{q} - \omega\dot{q}p - \omega q\dot{p} + \alpha\dot{p} + \sum_{n=1}^{M} \beta_n \sum_{k=1}^{n} q^{k-1}\dot{q}q^{n-k}$$

$$= 3q^3 + p^2 - \omega p^2 - 3\omega q^3 + 3\alpha q^2 + \sum_{n=1}^{M} \beta_n \sum_{k=1}^{n} q^{k-1}pq^{n-k} \in \mathfrak{J}$$

$$\Rightarrow \omega = 1, \alpha = \beta_n = 0$$

and therefore $J = pq - qp + \gamma$, where $\gamma \in \mathbb{C}$.

If we add the reality arguments in the consideration we would conclude that γ is pure imaginary, i.e. $\gamma = i\hbar$ and the Heisenberg quantisation is a unique possibility for the free associative mechanics (1).

In algebra $\mathfrak{A} = \mathbb{K}\langle u, v\rangle$ we consider systems of two quadratic homogeneous equations

$$\begin{cases} u_t = \alpha_1 u^2 + \alpha_2 uv + \alpha_3 vu + \alpha_4 v^2, \\ v_t = \beta_1 v^2 + \beta_2 vu + \beta_3 uv + \beta_4 u^2 \end{cases} \tag{2}$$

possesing a hierarchy of symmetries.

Let us first consider equations (2) possesing a cubic symmetry

$$\begin{cases} u_\tau = \gamma_1 u^3 + \gamma_2 u^2 v + \gamma_3 uvu + \gamma_4 vu^2 + \gamma_5 uv^2 + \gamma_6 vuv + \gamma_7 v^2 u + \gamma_8 v^3, \\ v_\tau = \delta_1 u^3 + \delta_2 u^2 v + \delta_3 uvu + \delta_4 vu^2 + \delta_5 uv^2 + \delta_6 vuv + \delta_7 v^2 u + \delta_8 v^3 \end{cases} \tag{3}$$

In $\mathfrak{A} = \mathbb{K}\langle u, v\rangle$ let us consider quantisation ideals generated by one polynomial

$$\mathfrak{J} = \langle vu - \alpha uv - \delta u^2 - \beta u - \gamma v - \eta \rangle$$

PROPOSITION

Any non-triangular system (2) possessing a non-zero cubic symmetry of the form (3) is equivalent to one of the following systems which admits a quantisation ideal $\mathfrak{J}$ generated by the comutation relation:

$$\mathrm{A}_1: \quad \begin{cases} u_t = u^2 - uv \\ v_t = v^2 + vu - uv \end{cases} \qquad uv = vu,$$

$$\mathrm{A}_2: \quad \begin{cases} u_t = uv \\ v_t = vu \end{cases} \qquad vu = \alpha uv, \qquad H = \alpha u - v,$$

$$\mathrm{A}_3: \quad \begin{cases} u_t = u^2 - uv \\ v_t = v^2 - vu \end{cases} \qquad vu = uv - \gamma u + \gamma v, \qquad H = uv - \gamma u$$

$$\mathrm{A}_4: \quad \begin{cases} u_t = -uv \\ v_t = v^2 + uv - vu \end{cases} \qquad vu = uv - \gamma v, \qquad H = uv + \gamma u$$

$$\mathrm{A}_5: \quad \begin{cases} u_t = uv - vu \\ v_t = u^2 + uv - vu \end{cases} \qquad vu = uv + \delta u^2 + \beta u + \eta, \quad \ldots$$

$$\mathrm{A}_6: \quad \begin{cases} u_t = v^2 \\ v_t = u^2 \end{cases} \qquad vu = uv + \eta, \qquad H = (v^3 - u^3)/3,$$

where $\alpha, \beta, \gamma, \delta, \eta \in \mathbb{K}$ are arbitrary constants and $\alpha \neq 0$ and H is the Hamiltonian.

$$\mathrm{A}_6: u_t = \eta^{-1}[H, u] = v^2, \quad v_t = \eta^{-1}[H, v] = u^2, \quad H_t = 0.$$

PROPOSITION

Any non-triangular system (2) possessing a symmetry of degree four, but not of a qubic one admit $\mathfrak{I}$ quantisation with the following commutation relations:

$$\mathrm{B}_1 \quad \begin{cases} u_t = -uv \\ v_t = v^2 + vu \end{cases} \qquad uv = vu + \gamma v, \quad H = 2uv + u^2 + \gamma u$$

$$\mathrm{B}_2 \quad \begin{cases} u_t = -vu \\ v_t = v^2 + vu \end{cases} \qquad uv = vu + \gamma v, \quad H = 2uv + u^2 + \gamma u + 2\gamma v$$

$$\mathrm{B}_3 \quad \begin{cases} u_t = u^2 - 2vu \\ v_t = v^2 - 2vu \end{cases} \qquad vu = uv + \eta, \quad H = u^2 v - uv^2,$$

$$\mathrm{B}_4 \quad \begin{cases} u_t = u^2 - uv - 2vu \\ v_t = v^2 - 2uv - vu \end{cases} \qquad vu = uv,$$

$$\mathrm{B}_5 \quad \begin{cases} u_t = u^2 - 2uv \\ v_t = v^2 + 4vu \end{cases} \qquad vu = uv,$$

where $\gamma, \eta \in \mathbb{K}$ are arbitrary constants.

Heisenberg equations:

$$\kappa u_t = [H, u], \qquad \kappa v_t = [H, v].$$

$$B_1 : \ \kappa = -2\gamma, \quad B_2 : \ \kappa = -2\gamma, \quad B_3 : \ \kappa = \eta.$$

QUANTISATION OF NOVIKOV'S SYSTEMS

Non-abelian KdV hierarchy:

$$L = D^2 - u, \quad L_{t_{2k+1}} = -u_{t_{2k+1}} = [\left(L^{\frac{2k+1}{2}}\right)_+, L] = -2D\mathrm{res}\left(L^{\frac{2k+1}{2}}\right), \quad D = \partial_x$$

Notations: $D(u) = u' = u_1,\ D^2(u) = u'' = u_2, \dots$

$$\begin{aligned}
u_{t_1} &= D(u) = u_1, \\
4u_{t_3} &= D(u_2 - 3u^2), \\
16u_{t_5} &= D(u_4 - 5u_2 u - 5uu_2 - 5u_1^2 + 10u^3), \\
64u_{t_7} &= D(u_6 - 7u \cdot u_4 - 14u_1 \cdot u_3 - 21u_2^2 - 14u_3 \cdot u_1 - 7u_4 \cdot u \\
&+ 21u^2 \cdot u_2 + 28u \cdot u_1^2 + 28u \cdot u_2 \cdot u + 14u_1 \cdot u \cdot u_1 \\
&+ 28u_1^2 \cdot u + 21u_2 \cdot u^2 - 35u^4), \\
2^{2k} u_{t_{2k+1}} &= D(u_{2k} + \cdots) = D(G_{2k+2}).
\end{aligned}$$

Non-abelian Novikov's equation: choose $N \in \mathbb{N}$

$$G_{2N+2} + \sum_{k=1}^{N-1} \alpha_{2(N+1-k)} G_{2k} = \alpha, \qquad \alpha, \alpha_k \in \mathbb{C}.$$

$$\begin{aligned}
N = 1: &\qquad u_2 - 3u^2 = \alpha \\
N = 2: &\qquad u_4 - 5u_2 u - 5uu_2 - 5u_1^2 + \beta_4 u = \alpha
\end{aligned}$$

Quantisation [V.Buchstaber, AVM]:

$$N=1:\quad [u_1,u]=\eta,\ \eta\partial_{t_1}(u)=[H,u]=\eta u_1,\ \eta\partial_{t_1}(u_1)=[H,u_1]=\eta(3u^2+\alpha),$$
$$H=\tfrac{1}{2}u_1^2-u^3-\alpha u,$$

$$N=2:\quad [u_1,u]=[u,u_2]=[u_1,u_3]=0,\ [u_3,u]=[u_1,u_2]=\eta,\ [u_3,u_2]=10\eta u;$$

$$\eta\partial_{t_1}(u)=[H_8,u]=\eta u_1,\ \eta\partial_{t_1}(u_1)=[H_8,u_1]=\eta u_2,\ \eta\partial_{t_1}(u_2)=[H_8,u_2]=\eta u_3,$$
$$\eta\partial_{t_1}(u_3)=[H_8,u_3]=\eta(\alpha-\alpha_4 u+5u_1^2+10u_2\cdot u-10u^3),$$

$$\eta\partial_{t_3}(u)=[H_{10},u]=\eta(u_3-6u_1u).$$
$$\eta\partial_{t_3}(u_k)=[H_{10},u_k]=\eta D^{k-1}(u_3-6u_1u).$$

$$2H_8 = 2u_3\cdot u_1-u_2^2-10u_1^2\cdot u+5u^4+\alpha_4u^2-2\alpha u$$

$$\begin{aligned}2H_{10} &= 8\eta u_1-6\alpha u^2+2\alpha u_2+\alpha_4u_1^2-2\alpha_4u_2\cdot u+4\alpha_4u^3\\ &- u_3^2-2u_2\cdot u_1^2+4u_2^2\cdot u+12u_3\cdot u_1\cdot u-30u_1^2\cdot u^2\\ &- 20u_2\cdot u^3+24u^5,\end{aligned}$$

$$[H_8,H_{10}]=0.$$

$$\dagger:\quad u_k^\dagger=u_k,\quad (ab)^\dagger=b^\dagger a^\dagger,\quad \eta^\dagger=-\eta,\quad \alpha^\dagger=\alpha,\quad \alpha_4^\dagger=\alpha_4,$$
$$H_8^\dagger=H_8,\qquad H_{10}^\dagger=H_{10}.$$

For $N \in \mathbb{N}$

$$F_{2n} = \varrho_{2n} + \sum_{k=0}^{n-1} \alpha_{2(n-k+1)} \varrho_{2k}, \qquad n = 1, 2, \ldots, \quad \varrho_{2n} = \mathrm{res} L^{\frac{2n-1}{2}}$$

$$F_{2N+2} = 0, \qquad \partial_{\tau_{2n-1}}(u) = -2D(F_{2n}), \quad n = 2, 3, \ldots, N.$$

In the commutative case, in the variables

$$v_{2n} = -F_{2n}, \qquad v_{2n+1} = -\frac{1}{2} D(F_{2n}), \qquad n = 1, 2, \ldots, N \tag{4}$$

the Poisson brackets takes the form

$$\{a, b\} = \sum_{n=1}^{N} \left(\frac{\partial a}{\partial v_{2N+3-2n}} \frac{\partial b}{\partial v_{2n}} - \frac{\partial b}{\partial v_{2N+3-2n}} \frac{\partial a}{\partial v_{2n}} \right) \tag{5}$$
$$- \sum_{n=1}^{N-1} \sum_{m=1}^{N-n} v_{2m} \left(\frac{\partial a}{\partial v_{2N+3-2n}} \frac{\partial b}{\partial v_{2m+2n}} - \frac{\partial b}{\partial v_{2N+3-2n}} \frac{\partial a}{\partial v_{2m+2n}} \right).$$

and ther is an invertible polynomial change of variables

$$p_n = v_{2N-2n+3}, \quad q_n = v_{2n} + V(v_2, \ldots, v_{2n-2})$$

$$\{a, b\} = \sum_{n=1}^{N} \left(\frac{\partial a}{\partial p_n} \frac{\partial b}{\partial q_n} - \frac{\partial a}{\partial q_n} \frac{\partial b}{\partial p_n} \right),$$

The same can be done at the quantum level:
For $N = 1$:

$$q_1 = Q_1 = v_2 = -F_2 = \frac{1}{2}u,\ p_1 = -\frac{1}{4}u_1 \ \Rightarrow\ [q_1, p_1] = \frac{1}{8}\eta.$$

Case $N = 2$ we have

$$Q_1 = v_2 = -F_2 = \frac{1}{2}u,\ Q_2 = v_4 = -F_4 = -\varrho_4 - \alpha_4\varrho_0 = \frac{1}{8}(u_2 - 3u^2) - \alpha_4$$

Thus

$$q_1 = Q_1 = \frac{1}{2}u, \qquad p_1 = v_5 = \frac{1}{2}D(v_4) = \frac{1}{16}(u_3 - 3uu_1 - 3u_1u),$$

$$q_2 = Q_2 + \frac{1}{2}Q_1^2 = \frac{1}{8}(u_2 - 2u^2) - \alpha_4, \qquad p_2 = v_3 = \frac{1}{2}D(v_2) = \frac{1}{4}u_1$$

and

$$[q_i, p_j] = \frac{\eta}{32}\delta_{i,j}, \qquad [p_i, p_j] = [q_i, q_j] = 0.$$

Let $\eta = 32i\hbar$, then $i\hbar(q_k)_{\tau_{2n-1}} = [q_k, H_n],\ i\hbar(p_k)_{\tau_{2n-1}} = [p_k, H_n]$

$$H_1 = -q_1p_2^2 - 2p_2p_1 + \alpha_4q_1^2 + 2\alpha_4q_2 + 2\alpha_6q_1 + q_2^2 + 2q_2q_1^2 - \tfrac{1}{4}q_1^4$$

$$H_2 = q_2p_2^2 - \tfrac{1}{2}q_1^2p_2^2 + p_1^2 - 2\alpha_4q_2q_1 + \alpha_4q_1^3 + \alpha_6q_1^2 - 2\alpha_6q_2 - 2q_2^2q_1 + q_2q_1^3 - i\hbar p_2,$$

$$[H_1, H_2] = 0, \quad H_1 = H_1^\dagger,\ H_2 = H_2^\dagger, \qquad (p_k \to -i\hbar\tfrac{d}{dq_k}).$$

Case $N = 3$ we have

$$q_1 = \tfrac{1}{2}u,$$
$$q_2 = \tfrac{1}{8}(u_2 - 2u^2) - \alpha_4,$$
$$q_3 = \tfrac{1}{96}(3u_4 - 15u_1^2 - 24u_2u + 16u^3) - \alpha_6,$$

$$v_7 = p_1 = \tfrac{1}{64}(16\alpha_4 u_1 - 20u_2u_1 - 10u_3u + 30u_1u^2 + u_5),$$
$$v_5 = p_2 = \tfrac{1}{16}(-3uu_1 - 3u_1u + u_3),$$
$$v_3 = p_3 = \tfrac{1}{4}u_1,$$

and ($[u, u_5] = \eta$)

$$[q_i, p_j] = \frac{\eta}{128}\delta_{i,j}, \qquad [p_i, p_j] = [q_i, q_j] = 0.$$

Let $\eta = 128i\hbar$. There are $H_1,\ H_2,\ H_3$

$$[H_1,\ H_2] = [H_1,\ H_3] = [H_2,\ H_3] = 0, \quad (p_k)_{\tau_{2n-1}} = \frac{i}{\hbar}[H_n, p_k],\ (q_k)_{\tau_{2n-1}} = \frac{i}{\hbar}[H_n, q_k].$$

For example:

$$H_1 = 1/2(H_{3,7} + H_{3,7}^{\dagger}),$$

$$H_{3,7} = \tfrac{3}{2^9}(14u^5 + 32\alpha_4u^3 - 64\alpha_6u^2 - 70u_1^2u^2 + 256\alpha_8u$$

$$-16\alpha_4u_1^2 - 14u_2^2u + 28u_3u_1u - u_3^2 + 28u_2u_1^2 + 2u_4u_2 - 2u_5u_1).$$

Let us consider quantisation of the stationary KdV equation:

$$4u_{t_3} = \partial_{t_1}(u_2 - 3u^2) = u_3 - 3u_1 u - 3uu_1 = 0 \;\Rightarrow\; \mathcal{I} = \langle u_3 - 3u_1 u - 3uu_1 \rangle$$

Then $\mathfrak{A} \diagup \mathcal{I} = \mathbb{C}\langle u, u_1, u_2 \rangle$ and

$$\partial_{t_1}(u) = u_1, \quad \partial_{t_1}(u_1) = u_2, \quad \partial_{t_1}(u_2) = 3u_1 u + 3uu_1.$$

Let us consider a general homogeneous differential ideal generated by

$$\mathfrak{I} = \langle [u, u_1] - \alpha u^2 - \beta u_1 - \gamma u - \delta \rangle$$

PROPOSITION

The ideal $\mathfrak{I} = \langle [u, u_1] - \alpha u^2 - \beta u_1 - \gamma u - \delta \rangle$ is ∂_{t_1} invariant if and only if $\alpha = \beta = 0$, and thus

$$[u, u_1] - \delta - \gamma u, \quad [u, u_2] - \gamma u_1, \quad [u_1, u_2] + \gamma(6u^2 - u_2) + 6\delta u.$$

The dynamical system (the stationary KdV equation)

$$\partial_{t_1} u = u_1, \quad \partial_{t_1} u_1 = u_2, \quad \partial_{t_1} u_2 = 3u_1 u + 3uu_1$$

has TWO compatible quantisations

$$\begin{array}{llll} (A): & [u, u_1]_\delta = \delta, & [u, u_2]_\delta = 0, & [u_2, u_1]_\delta = -6\delta u \\ (B): & [u, u_1]_\gamma = \gamma u, & [u, u_2]_\gamma = \gamma u_1, & [u_2, u_1]_\gamma = \gamma(6u^2 - u_2) \end{array}$$

It has two Hamiltonians

$$(1): \quad H_0 = u_2 - 3u^2, \qquad (2): \quad H_1 = \frac{1}{2}(u_1^2 - u_2 u - uu_2) + 2u^3$$

H_0 is Casimir for δ and H_1 is Casimir for γ comutation relations

$$[u_k, H_0]_\delta = 0, \qquad [u_k, H_1]_\gamma = 0, \qquad k = 0, 1, 2.$$

and the dynamical system can be written in the Heisenberg form:

$$\partial_{t_1} u_0 = -\tfrac{1}{\gamma}[H_0, u_0]_\gamma = -\tfrac{1}{\delta}[H_1, u_0]_\delta = u_1,$$

$$\partial_{t_1} u_1 = -\tfrac{1}{\gamma}[H_0, u_1]_\gamma = -\tfrac{1}{\delta}[H_1, u_1]_\delta = u_2,$$

$$\partial_{t_1} u_2 = -\tfrac{1}{\gamma}[H_0, u_2]_\gamma = -\tfrac{1}{\delta}[H_1, u_2]_\delta = 3u_1 u + 3uu_1.$$

We have : $3u_1 u + 3uu_1 :_\delta = 6u_1 u + 3\delta$, $: 3u_1 u + 3uu_1 :_\gamma = 6u_1 u + 3\gamma u$.

BI-QUANTISATION OF THE $N = 1$ NOVIKOV EQUATION

$$\partial_{t_1}(u_2 - 3u^2) = 0 \;\Rightarrow u_2 = 3u^2 + \hat{\alpha}, \quad \text{where } \hat{\alpha} \text{ is a non-commuative constant.}$$

The invariant $(\partial_{t_1} u = u_1,\ \partial_{t_1} u_1 = 3u^2 + \hat{\alpha},\ \partial_{t_1}\hat{\alpha} = 0)$ ideal in $\mathbb{C}\langle u, u_1, \hat{\alpha}\rangle$ is generated by

$$[u, u_1] = \gamma u + \delta, \quad [u, \hat{\alpha}] = \gamma u_1, \quad [u_1, \hat{\alpha}] = 3\gamma u^2 + \gamma\hat{\alpha}.$$

Thus there are two compatible quantisations:

$$\begin{array}{llll} (A): & [u, u_1]_\delta = \delta, & [u, \hat{\alpha}]_\delta = 0, & [u_1, \hat{\alpha}]_\delta = 0 \\ (B): & [u, u_1]_\gamma = \gamma u, & [u, \hat{\alpha}]_\gamma = \gamma u_1, & [u_1, \hat{\alpha}]_\gamma = \gamma(3u^2 + \hat{\alpha}) \end{array}$$

There are two Hamiltonians

$$H_0 = \hat{\alpha}, \qquad H_1 = \frac{1}{2}(u_1^2 - 4u^3 - u\hat{\alpha} - \hat{\alpha}u), \qquad [H_0, H_1] = 0,$$

such that $[H_0, \cdot]_\delta = 0,\ [H_1, \cdot]_\gamma = 0$ and

$$\begin{aligned} \partial_{t_1} u \; &= -\tfrac{1}{\gamma}[H_0, u\ ]_\gamma = -\tfrac{1}{\delta}[H_1, u\ ]_\delta = u_1, \\ \partial_{t_1} u_1 &= -\tfrac{1}{\gamma}[H_0, u_1]_\gamma = -\tfrac{1}{\delta}[H_1, u_1]_\delta = 3u^2 + \hat{\alpha}, \\ \partial_{t_1} \hat{\alpha} \; &= -\tfrac{1}{\gamma}[H_0, \hat{\alpha}\ ]_\gamma = -\tfrac{1}{\delta}[H_1, \hat{\alpha}\ ]_\delta = 0. \end{aligned}$$

In general, for the N-th Novikov equation as well as for stationary $2N+1$ KdV equation we have obtained a bi-quantum structure of the form:
In each case there are

- two compatible commutation rules $[\cdot,\cdot]_\delta,\ [\cdot,\cdot]_\gamma$;
- $N+1$ commuting operators $H_0, H_1, \ldots, H_N$;
- the finite quantum hierarchy of commuting symmetries which can be presented in the Heisenberg form

$$\begin{array}{rcl} & & [H_0, u_k]_\delta = 0, \\ u_{k,t_1} & = & \delta^{-1}[H_1, u_k]_\delta = \gamma^{-1}[H_0, u_k]_\gamma, \\ u_{k,t_3} & = & \delta^{-1}[H_2, u_k]_\delta = \gamma^{-1}[H_1, u_k]_\gamma, \\ & & \cdots \\ u_{k,t_{2N-1}} & = & \delta^{-1}[H_N, u_k]_\delta = \gamma^{-1}[H_{N-1}, u_k]_\gamma, \\ & & [H_N, u_k]_\gamma = 0. \end{array} , \qquad k = 0, 1, \ldots 2N-1, 2N.$$

Quantisation of Volterra and Bogoyavlensky systems

Let us consider nonabelian integrable systems: the Volterra chain (i) and the Bogoyavlensky N–chains (ii)

$$\text{(i) } \partial_{t_2} u = u_1 u - u u_{-1}, \quad \text{(ii) } \partial_{t_2} u = \sum_{k=1}^{N} (u_k u - u u_{-k}). \tag{6}$$

These are infinite systems of equations.
We use standard notations $u = u_0 = u(n,t),\ u_k = u_k(n,t) = u(n+k,t),\ n,k \in \mathbb{Z}$.

In equations (6) functions u_k are elements of a free associative algebra $\mathfrak{A} = \mathbb{C}\langle \dots u_{-1}, u, u_1, \dots \rangle$ with an infinite number of variables u_k and a natural automorphism $S : \mathfrak{A} \mapsto \mathfrak{A}$, generated by the shift operator $S(u_k) = u_{k+1},\ k \in \mathbb{Z}$, and $\partial_{t_2} S = S \partial_{t_2}$.

We begin with consideration of two-sided ideals $\mathfrak{J}_\omega \subset \mathfrak{A}$ generated by an infinite set of polynomials of the form

$$\mathfrak{J}_\omega = \langle \{ u_q u_p - \omega_{p,q} u_p u_q \,|\, p,q \in \mathbb{Z},\ p > q,\ \omega_{p,q} \in \mathbb{C}^\times \} \rangle$$

$$u_p u_q = \omega_{q,p} u_q u_p, \qquad q > p, \quad \omega_{p,q} \neq 0.$$

QUANTISATION OF THE VOLTERRA SYSTEMS

PROPOSITION

Volterra system $\partial_{t_2} u = u_1 u - u u_{-1}$ *can be restricted to* $\mathfrak{A}_{\mathfrak{I}_\omega}$ *if and only if* $\omega_{n+1,n} = \alpha,\ \omega_{n,m} = 1,\ n - m \geq 2$.

$$u_n u_{n+1} = \alpha u_{n+1} u_n, \quad u_n u_m = u_m u_n, \ \ |n - m| \geq 2. \tag{7}$$

The non-abelian Volterra system has a symmetry

$$u_{t_3} = u u_{-1} u_{-2} + u u_{-1} u_{-1} + u u u_{-1} - u_1 u u - u_1 u_1 u - u_2 u_1 u. \tag{8}$$

PROPOSITION

Equation (8) *can be restricted to* $\mathfrak{A}_{\mathfrak{I}_\omega}$ *only in the following cases:*

$$\text{(a)} \qquad u_n u_{n+1} = \alpha u_{n+1} u_n, \quad u_n u_m = u_m u_n, \quad |n - m| \geq 2 \tag{9}$$

$$\text{(b)} \qquad u_n u_{n+1} = (-1)^n \alpha u_{n+1} u_n, \quad u_n u_m = -u_m u_n, \quad |n - m| \geq 2 \tag{10}$$

PROPOSITION (S.CARPENTIER, AVM, J.P.WANG)

Every equation from the Volterra hierarchy admits quantisation (7).
Every odd degree equation from the Volterra hierarchy admits quantisation (10).

Partial quantisation (?!) The "cubic" difference ideal generated by polynomials

$$u_m u_n - u_n u_m, \ \ |m - n| > 1, \qquad u_{n+1} u_n u_{n+2} - u_n u_{n+2} u_{n+1}$$

is invariant w.r.t. the Volterra system and its symmetries.

Periodic closures of the chains $u_{k+M} = u_k$ with period M result in nonabelian systems on $\mathfrak{A}^M = \mathbb{C}\langle u_1, \ldots, u_M\rangle$.

Let $M = 3$:

$$\begin{aligned} u_{1,t_2} &= u_2 u_1 - u_1 u_3, \\ u_{2,t_2} &= u_3 u_2 - u_2 u_1, \\ u_{3,t_2} &= u_1 u_3 - u_3 u_2 \end{aligned}$$

The $M = 3$ Volterra system has an obvious constant of motion $H = u_1 + u_2 + u_3$.

It has infinitely many commuting symmetries:

$$\begin{aligned} u_{1,t_3} \quad &= u_1^2 u_3 + u_1 u_3 u_2 + u_1 u_3^2 - u_2 u_1^2 - u_2^2 u_1 - u_3 u_2 u_1, \\[1em] u_{1,t_4} \quad &= u_1^3 u_3 + u_1^2 u_3 u_2 + u_1^2 u_3^2 + u_1 u_2 u_1 u_3 + u_1 u_3 u_1 u_3 + u_1 u_3 u_2^2 \\ &\quad + u_1 u_3 u_2 u_3 + u_1 u_3^2 u_2 + u_1 u_3^3 - u_2 u_1^3 - u_2 u_1 u_2 u_1 - u_2 u_1 u_3 u_1 \\ &\quad - u_2^2 u_1^2 - u_2^3 u_1 - u_2 u_3 u_2 u_1 - u_3 u_2 u_1^2 - u_3 u_2^2 u_1 - u_3^2 u_2 u_1 \\ \cdots \quad &= \cdots \end{aligned}$$

QUANTISATION OF THE PERIODIC VOLTERRA CHAINS

Periodic Volterra systems with period M may admit inhomogeneous commutation relations:

$$u_q u_p = \omega_{p,q} u_p u_q + \sum_{r=1}^{M} \sigma_{p,q}^{r} u_r + \eta_{p,q}, \qquad 1 \leq q < p \leq M, \;\; \omega_{p,q} \neq 0.$$

PROPOSITION

Nonabelian periodical Volterra chain with period M admits $\mathfrak{I}_M$–quantisation iff the following commutation relations

$$\begin{aligned}
M = 3: \quad & u_n u_{n+1} = \alpha u_{n+1} u_n + \beta(u + u_1 + u_2) + \eta, \;\; n \in \mathbb{Z}_3;\\
M = 4: \quad & u_1 u_2 = \alpha u_2 u_1 + \beta u_2 + \gamma u_1 - \beta\gamma,\\
& u_1 u_3 = u_3 u_1 - \beta u_2 + \beta u_4,\\
& u_4 u_1 = \alpha u_1 u_4 + \beta u_4 + \gamma u_1 - \beta\gamma,\\
& u_2 u_3 = \alpha u_3 u_2 + \beta u_2 + \gamma u_3 - \beta\gamma,\\
& u_2 u_4 = u_4 u_2 - \gamma u_3 + \gamma u_1,\\
& u_3 u_4 = \alpha u_4 u_3 + \beta u_4 + \gamma u_3 - \beta\gamma;\\
M \geq 5: \quad & u_{n+1} u_n = \alpha u_n u_{n+1},\\
& u_n u_m = u_m u_n, \;\; |n - m| > 1, \; n, m \in \mathbb{Z}_M.
\end{aligned}$$

take place. The constants $\alpha, \beta, \gamma, \eta \in \mathbb{C}, \; \alpha \neq 0$ are arbitrary.

In the case $M = 3$ the ideal of quantisation has three parameters of quantisation

$$\mathfrak{J}_3 = \langle u_n u_{n+1} - \alpha u_{n+1} u_n - \beta(u + u_1 + u_2) - \eta \mid n \in \mathbb{Z}_3,\ \alpha \neq 0\rangle.$$

On $\mathfrak{A}_{\mathfrak{J}_3}$ the quantum Volterra system can be written as

$$u_{k,t_2} = \frac{1}{1-\alpha}[H, u_k], \qquad H = u_1 + u_2 + u_3.$$

The symmetry ∂_{τ_1} on $\mathfrak{A}_{\mathfrak{J}_3}$ is not independent:

$$u_{k,t_3} = \frac{1}{1+\alpha}\left(u_{k,t_2} H + H u_{k,t_2} + \frac{2(\alpha-1)\beta}{\alpha+1} u_{k,t_2}\right).$$

The system admits a Casimir operator ($[\mathcal{C}, u_k] = 0,\ k \in \mathbb{Z}_3$):

$$\begin{aligned}
\mathcal{H} \quad &= (\alpha^2 - 1) u_3 u_2 u_1 + (\alpha\beta + \beta)(u_2 u_1 + u_3 u_1 + u_3 u_2) \\
&+ \alpha\beta u_1^2 + \alpha^{-1}\beta u_2^2 + \alpha\beta u_3^2 \\
&+ u_1(\alpha\eta + \beta^2 + \eta) + u_2(\alpha^{-1}\eta + \beta^2 + \eta) + u_3(\alpha\eta + \beta^2 + \eta).
\end{aligned}$$

Quantisation (b) of odd degree periodic Volterra symmetries

The minimal period is $M = 4$. The quantisation (b) ideal $\mathfrak{J}_b$ is generated by:

$$
\begin{aligned}
&uu_1 = \alpha u_1 u, && uu_2 = -u_2 u, && uu_3 = -u_3 u,\\
&u_1 u_2 = -\alpha u_2 u_1, && u_1 u_3 = -u_3 u_1, && u_2 u_3 = \alpha u_3 u_2.
\end{aligned}
$$

There is a first integral H_1 and two Casimir operators $H_4, \hat{H}_4$:

$$H_1 = u + u_1 + u_2 + u_3, \quad \mathcal{H}_4 = u_2^2 u^2 + u_3^2 u_1^2, \quad \hat{\mathcal{H}}_4 = u_3 u_2 u_1 u, \quad [H_4, \cdot] = [\hat{H}_4, \cdot] = 0.$$

On $\mathfrak{A}_4 \diagup \mathfrak{J}_b$ the quantum Volterra system can be written as

$$u_{k,t_3} = \frac{1}{\alpha^2 - 1}[H_1^2, u_k], \qquad k = 0, 1, 2, 3.$$

The next (fifth degree) symmetry ∂_{t_5} on $\mathfrak{A}_4 \diagup \mathfrak{J}_b$ is dependent :

$$u_{k,t_5} = \frac{1}{1 + \alpha^2}\left(u_{k,t_3} H_1^2 + H_1^2 u_{k,t_3}\right).$$

In the general case ($N = 2n,\ n \geqslant 2$) there are:

- $n - 1$ commuting Hamiltonians $H_2 = (\sum_{k=0}^{2n-1} u_k)^2,\ \ H_4, \ldots,\ \ H_{2n-2},\ \ [H_{2i}, H_{2j}] = 0,$
- $n - 1$ independent symmetries $\partial_{t_{2k+1}} u = (\alpha^2 - 1)^{-1}[H_{2k}, u],$
- Two Casimir operators

$$\mathcal{H}_{2n} = u_{2n-1}^2 u_{2n-3}^2 \cdots u_1^2 + u_{2n-2}^2 u_{2n-4}^2 \cdots u^2, \quad \hat{\mathcal{H}}_{2n} = u_{2n-1} u_{2n-2} \cdots u_1 u.$$

QUANTISATION OF THE BOGOYAVLENSKY FAMILY OF SYSTEMS

PROPOSITION

Nonabelian N–chain $\partial_t u = \sum_{k=1}^{N}(u_k u - u u_{-k})$ *admits* $\mathfrak{J}_\omega = \langle\{u_q u_p - \omega_{p,q} u_p u_q \mid p, q \in \mathbb{Z},\ p > q,\ \omega_{p,q} \in \mathbb{C}^\times\}\rangle$ *quantisation only in the case*

$$\omega_{n+k,n} = \alpha, \quad \textit{where } 1 \le k \le N,\ \alpha \neq 0,\ \textit{and} \quad \omega_{n,m} = 1, \quad \textit{for } n - m > N.$$

$$u_n u_{n+k} = \alpha u_{n+k} u_n, \ \ 1 \le k \le N \qquad u_n u_m = u_m u_n, \ \ |n - m| > N.$$

PROPOSITION

There exists a modification $(u = v_2 v_1 v)$

$$v_t = v_2 v_1 v^2 + v_1 v v_{-1} v - v v_1 v v_{-1} - v^2 v_{-1} v_{-2}$$

of the nonabelian $N = 2$ *Bogoyavlensky chain. It admits* $\mathfrak{J}_\omega$*–quantisation only in the case*

$$\omega_{n+3m+1,n} = \alpha, \quad \omega_{n+3m+2,n} = \beta, \quad \omega_{n+3m+3,n} = \alpha^{-1}\beta^{-1},$$

$\alpha, \beta \in \mathbb{C}^\times,\ n \in \mathbb{Z},\ m \in \mathbb{Z}_{\geqslant 0}$

$$v_n v_{n+3m+1} = \alpha v_{n+3m+1} v_n,\ \ v_n v_{n+3m+2} = \beta v_{n+3m+2} v_n,\ \ v_n v_{n+3m+3} = \alpha^{-1}\beta^{-1} v_{n+3m+3} v_n,\ m \geqslant 0$$

$$u = v_2 v_1 v \ \ \Rightarrow \ \ u_n u_{n+1} = \alpha\beta u_{n+1} u_n,\ u_n u_{n+2} = \alpha\beta u_{n+2} u_n,\ u_n u_m = u_m u_n \text{ if } |m - n| > 2.$$