

Algebra of the infrared and secondary polytopes

M. Kapranov, M. Kontsevich, Y. Soibelman

August 13, 2014

To the memory of Andrei Zelevinsky

Contents

0	Introduction	3
1	Reminder on secondary polytopes.	7
2	A commutative dg-algebra from the chain complex of $\Sigma(A)$.	14
3	The L_∞ -algebra.	16
4	Maurer-Cartan elements in \mathfrak{g} .	21
5	1-dimensional case: refinement to an A_∞ -algebra.	23
6	Relative setting: one point “at infinity”.	25
7	Hochschild complexes and derived derivation spaces.	27
8	Refinement in $d = 2$: relative setting.	33

9	Introducing coefficients: factorizing sheaves on secondary polytopes.	35
10	Coefficients in bimodules: $d = 2$	38
11	Analysis of the algebra R_∞ and the morphism ψ for $d = 2$.	44
12	The universality theorem ($d = 2$).	49
13	Comparison with Gaiotto-Moore-Witten.	57
14	Maurer-Cartan elements for the Fukaya-Seidel category	61
15	Speculations and directions for further work	72

0 Introduction

The words “algebra of the infrared” in the title refer to the physical paper [GMW] by Gaiotto, Moore and Witten, to which (or, rather, to a part of which) our article is a mathematical commentary.

In [GMW] the authors developed an algebraic formalism for the study of certain 2-dimensional massive quantum field theories with $(2, 2)$ supersymmetry. One of the main algebraic structures introduced in the *loc.cit.* is the L_∞ -algebra of webs, associated with a generic finite subset $A \subset \mathbf{R}^2$ in the plane. Physically, elements of A correspond to vacua of the theory. A web is a plane graph with faces marked by elements of A with an additional condition on the direction of edges, see §13 below for a review. Further, a choice of a half-plane containing A determines an A_∞ -algebra (or an A_∞ -category, if one introduces a coefficient system). This A_∞ -category has an “upper-triangular structure”, i.e. a semi-orthogonal decomposition.

Using certain “moduli spaces of ζ -instantons” the authors of *loc.cit.* describe a class of deformations of the above A_∞ -category which describe the D-brane categories for a particular class of $(2,2)$ supersymmetric theories: Landau-Ginzburg models. Mathematically, the D-brane A_∞ -categories corresponding to LG models are known as Fukaya-Seidel categories [HIV],[Se1].

We reinterpret and develop the algebraic structures proposed in [GMW] in a way that allows a generalization to higher dimensions (\mathbf{R}^d , $d \geq 2$ instead of just \mathbf{R}^2). It turns out that using the dual language of polygons rather than webs, one quickly uncovers certain structures well-known in toric geometry, most notably, *secondary polytopes*, see [GKZ]. One of the subtle and surprising points of the GMW construction is the fact that the differential they define, satisfies $d^2 = 0$. In our approach this fact becomes obvious: the cellular chain complex of any polytope, in particular, of the secondary polytope, is indeed a complex, i.e., it has $d^2 = 0$.

Since secondary polytopes make sense in any number of dimensions, our dual approach leads naturally to higher-dimensional structures of “extended” topological field theories (TFT). We postpone the study of these structures to a later work. So the present paper is just the first step in that general direction.

Let us discuss the contents.

Although we were strongly influenced by [GMW], we have arranged the exposition logically from scratch, implementing our dual interpretation right

away. A detailed dictionary relating our approach and terminology with that of [GMW], is provided in Section 13. The reader familiar with the terminology of [GMW] can start from that section and then proceed linearly from the beginning.

Section 1 provides the general background on secondary polytopes. Roughly, the secondary polytope of a given polytope in \mathbf{R}^d has vertices corresponding to decompositions of the initial, “primary”, polytope into simplices (such decompositions are called triangulations). The usefulness of the notion of secondary polytope from the point of view of the approach of [GMW] becomes clear in Section 2: the chain complex of the secondary polytope looks like a dg-algebra. The main reason for that is explained earlier in Proposition 1.9: each face of the secondary polytope is itself a product of several secondary polytopes associated to members of a certain polyhedral subdivision of the initial polytope.

In Section 3 we introduce an L_∞ -algebra structure in the space \mathfrak{g}_A spanned by polytopes with vertices in a given set $A \subset \mathbf{R}^d$. For that we define a differential in the symmetric algebra generated by the (shifted) dual space to \mathfrak{g}_A . The remark in the previous paragraph about the chain complex of the secondary polytope explains why our L_∞ -structure comes naturally from such a chain complex. Further, one can visualize higher Lie brackets in \mathfrak{g}_A geometrically. Roughly speaking, they correspond to the operation of composing a convex polytope from smaller polytopes with vertices in A .

Section 4 is a reminder about Maurer-Cartan elements in L_∞ -algebras. Together with Section 7, which is devoted to a short account on deformation theory, Hochschild complexes etc., it prepares the ground for future applications à la [GMW]: a geometrically defined Maurer-Cartan element gives rise to a deformed L_∞ -structure.

Section 5 is devoted to the most elementary case $d = 1$, which explains the reason for the second construction of [GMW]: lifting of a certain L_∞ -algebra (related but not equal to \mathfrak{g}_A) to an A_∞ -algebra. The possibility of such lifting is the essential 1-dimensionality of the situation, and the purely 1-dimensional case provides a clear starting point. This is related to the fact that convex polytopes on the real line are just segments, and so one can naturally order their vertices.

In Section 6 we study the situation when one element of A is distinguished and referred to as an “element at infinity” (denoted ∞). This is a higher-dimensional analog of the choice of a half-plane in [GMW]. In this case the L_∞ -algebra \mathfrak{g}_A contains an ideal \mathfrak{g}_∞ spanned by “infinite” polytopes

(containing ∞), and a subalgebra $\mathfrak{g}_{\text{fin}}$ spanned by “finite” polytopes (not containing ∞). We explain in Section 8 how these structures lead, in the case $d = 2$, to an action of $\mathfrak{g}_{\text{fin}}$ on R_∞ , a natural A_∞ -lift of the L_∞ -algebra \mathfrak{g}_∞ . Combining this with the results of Sections 6, 7, we arrive at a morphism from $\mathfrak{g}_{\text{fin}}$ to the Hochschild complex of R_∞ (all in the case of trivial coefficients).

The case of non-trivial coefficients is the subject of Section 9. Here we introduce and exploit an important notion of a *factorizing sheaf* (on the secondary polytope). This notion can be seen as a combinatorial version of factorization algebras such as used by Lurie and Costello-Gwylliam [Lur, CGW] to describe higher-dimensional TFT. This strongly suggests that the constructions of the previous sections admit a generalization to the framework of higher-dimensional TFT, with, for instance, E_{d-1} -algebras playing the role of A_∞ -algebras such as R_∞ .

For an A_∞ -algebra with an upper-triangular structure, there is a version of the Hochschild complex which controls deformations preserving this structure. We call it the *directed Hochschild complex* and discuss it in Section . After preparations in Sections 10, 11 we finally prove, in Section 12 a result which we call the Universality Theorem. It concerns the case $d = 2$ and the situation of Section 11 associated with a choice of a point ∞ (or, equivalently, of a half-plane). It says that the natural morphism from the L_∞ -algebra $\mathfrak{g}_{\text{fin}}$ of finite polygons to the Hochschild complex of R_∞ , is a quasi-isomorphism onto the directed Hochschild complex.

This result should be compared with the Deligne conjecture, see [KoSo2], as well as with similar results about the Swiss Cheese operad and its higher-dimensional generalizations, see e.g., [Ko1]. It certainly admits a generalization to the higher-dimensional case (in which case one deals with Hochschild complexes of E_{d-1} -algebras in the spirit of [Ko1]). This will be a subject of separate publication.

In Section 14 we discuss possibilities for the Maurer-Cartan element giving the Fukaya-Seidel category associated with a choice of a half-plane.

Section 15 contains a list of further applications and speculations.

We should mention here that there are several L_∞ and A_∞ algebras that can be naturally associated with a finite subset $A \subset \mathbb{R}^d$. The biggest one involves all marked polytopes. It has non-trivial differential corresponding to the operation of “insertion of internal points”. On the other hand, for the application to Fukaya-Seidel categories one needs a much smaller L_∞ -subalgebra which involves polytopes, each marked with the set of all its internal points. The differential is trivial on this subalgebra, and moreover,

we do not need to include the set of points as a separate piece of the data. In this way we obtain a smaller L_∞ -algebra, which we refer to as the *geometric* one. It is spanned by convex polytopes with vertices in A , with higher Lie brackets given by the operation of composing a bigger convex polytope from smaller ones. As we pointed out above, the geometric L_∞ -algebra appears naturally in the description of the Hochschild complex of the Fukaya-Seidel category. The meaning of the “big” L_∞ -algebra is not clear at the moment.

Acknowledgements. We thank Davide Gaiotto, Greg Moore and Edward Witten for multiple discussions and correspondences about their deep and beautiful paper [GMW] as well as for sending us preliminary drafts. Y.S. is grateful to IHES for excellent research conditions. His work was partially supported by an NSF grant.

1 Reminder on secondary polytopes.

In this section we recall some basic material from [GKZ], Ch. 7. We refer the reader to *loc.cit.* for details of constructions and proofs of statements.

Let $A \subset \mathbb{R}^d$ be a finite subset of points. We assume that:

- (1) The affine span of A is the whole of \mathbb{R}^d .
- (2) A is in general position, i.e., any $p \leq d + 1$ points of A are the vertices of a $(p - 1)$ -simplex.

The first assumption can always be satisfied by passing to the affine span. The second assumption can be removed at the price of somewhat complicating the discussion and we prefer to keep it throughout the paper for simplicity.

Let $Q = \text{Conv}(A)$ be the convex hull of A , a d -dimensional convex polytope. By a *triangulation* \mathcal{T} of (Q, A) we will mean a subdivision of Q into straight simplices of full dimension d with vertices in A so that the intersection of any two simplices is a common face (possibly empty). Note that it is not required that each element of A appears as a vertex of a simplex of \mathcal{T} . A triangulation \mathcal{T} is called *regular*, if there is a continuous convex function $f : Q \rightarrow \mathbb{R}$ such that:

- f is affine-linear on each simplex of \mathcal{T} .
- f is not affine-linear on any subset of Q which is not contained in a simplex of \mathcal{T} . In other words, f does indeed break along each codimension 1 simplex which is a common face of two different d -dimensional simplices of \mathcal{T} .

For a more systematic discussion, we need the following concept.

Definition 1.1. A *marked polytope* is a pair (Q, A) where Q is a convex polytope in \mathbb{R}^d and $A \subset \mathbb{R}^d$ is a finite subset such that $Q = \text{Conv}(A)$, i.e., A contains all vertices of Q . A *marked subpolytope* of (Q, A) is a marked polytope (Q', A') such that $A' \subset A$. Notation: $(Q', A') \subset (Q, A)$.

Definition 1.2. By a *polyhedral subdivision* of (Q, A) we will mean a collection $\mathcal{P} = (Q_\nu, A_\nu)$ of marked subpolytopes of (Q, A) which have $\dim Q_\nu = \dim Q$ such that:

- (1) The Q_ν form a polyhedral subdivision of Q (i.e. $Q = \cup_\nu Q_\nu$) so that each $Q_\nu \cap Q_{\nu'}$ is a common face of Q_ν and $Q_{\nu'}$ (possibly empty).
- (2) In addition, we have $A_\nu \cap (Q_\nu \cap Q_{\nu'}) = A_{\nu'} \cap (Q_\nu \cap Q_{\nu'})$.

As for triangulations, a polyhedral subdivision $\mathcal{P} = (Q_\nu, A_\nu)$ is called *regular*, if there is a continuous convex function $f : Q \rightarrow \mathbb{R}$ which is affine-linear on each Q_ν and is not affine linear on any subset which is not contained in some Q_ν .

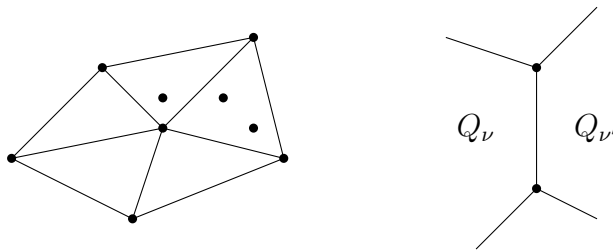


Figure 1: A triangulation and a polyhedral subdivision.

Regular polyhedral subdivisions of (Q, A) form a poset (partially ordered set) $\mathcal{R}(Q, A)$ with the order given by *refinement*. That is,

$$\mathcal{P}' = (Q'_\mu, A'_\mu) \leq \mathcal{P} = (Q_\nu, A_\nu),$$

iff \mathcal{P}' induces a (necessarily regular) polyhedral subdivision of each (Q_ν, A_ν) . Thus, minimal elements of $\mathcal{R}(Q, A)$ are precisely the regular triangulations, while the unique maximal element is the subdivision consisting of (Q, A) alone.

Remark 1.3. Note the essential role of the choices of markings A_ν in the definition of polyhedral subdivisions and of the poset $\mathcal{R}(Q, A)$. We do not require that $A = \cup_\nu A_\nu$. E.g. if (Q, A) is a triangle with two points in the interior then (Q', A') is a polyhedral subdivision provided $Q' = Q$ and A' is just one of the interior points.

More generally, if $A' \subset A$ is the set of vertices of Q , then each intermediate subset $A' \subset B \subset A$ gives rise to a 1-element regular polyhedral subdivision $(Q, B) \in \mathcal{R}(Q, A)$, and

$$(Q, B') \leq (Q, B) \text{ iff } B' \subset B.$$

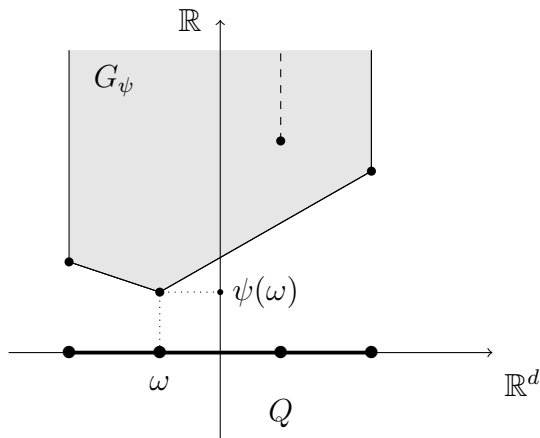


Figure 2: The polyhedron G_ψ and the function f_ψ .

The poset $\mathcal{R}(Q, A)$ has an interpretation in terms of the *secondary fan* of A . This fan, denoted $\mathcal{S}(A)$, is a subdivision of \mathbb{R}^A into convex polyhedral cones $C_{\mathcal{P}}$ corresponding to regular polyhedral subdivisions $\mathcal{P} \in \mathcal{R}(Q, A)$. To define it, we associate to each $\psi : A \rightarrow \mathbb{R}$ a convex piecewise-affine function $f_\psi : Q \rightarrow \mathbb{R}$ as follows. We consider the unbounded polyhedron $G_\psi \subset \mathbb{R}^{d+1} = \mathbb{R} \times \mathbb{R}^d$ given by

$$G_\psi = \text{Conv}\{(t, a) \mid t \in \mathbb{R}, a \in A \subset \mathbb{R}^d, t \geq \psi(a)\}.$$

This image of G_ψ under the projection to \mathbb{R}^d is Q , and the bottom of G_ψ is the graph of a convex piecewise-affine function $f_\psi : Q \rightarrow \mathbb{R}$. The function f_ψ gives rise to unique (necessarily regular) polyhedral subdivision $\mathcal{P}(\psi)$ consisting of *domains of affine linearity* of f_ψ . The cone $C_{\mathcal{P}} \subset \mathbb{R}^A$ is, by definition, the set of all ψ such that $\mathcal{P}_\psi = \mathcal{P}$.

Note that each $C_{\mathcal{P}}$ is invariant under translations by those ψ which come from global affine linear functions on \mathbb{R}^d . We denote the space of such functions by $\text{Aff}(\mathbb{R}^d)$. Then $C_{\mathcal{P}}$ gives rise to the cone $\overline{C}_{\mathcal{P}}$ in the quotient space $\mathbb{R}^A / \text{Aff}(\mathbb{R}^d)$. The collection of these cones is called the *reduced secondary fan* of A and denoted by $\overline{\mathcal{S}}(A)$.

Proposition 1.4. *The correspondence $\mathcal{P} \mapsto C_{\mathcal{P}}$ is an order reversing isomorphism between the poset $\mathcal{R}(Q, A)$ and the poset of cones of $\mathcal{S}(A)$ (ordered by inclusion). \square*

Along with $\mathcal{S}(A)$ we will consider the dual object, the *secondary polytope* $\Sigma(A)$ whose faces correspond to subdivisions $\mathcal{P} \in \mathcal{R}(Q, A)$ in an order-preserving, not reversing way. The construction of $\Sigma(A)$ depends on the choice of a translation invariant measure Vol on \mathbb{R}^d , a different choice leading to a rescaling of $\Sigma(A)$. For convenience of the reader we recall two (equivalent) definitions.

Definition 1.5. For a regular triangulation \mathcal{T} of (Q, A) define the vector $\phi_{\mathcal{T}} \in \mathbb{R}^A$ by

$$\phi_{\mathcal{T}}(\omega) = \sum_{\substack{\Delta \in \mathcal{T} \\ \omega \in \text{Vert}(\Delta)}} \text{Vol}(\Delta).$$

The polytope $\Sigma(A) \subset \mathbb{R}^A$ is defined to be the convex hull of the vectors $\phi_{\mathcal{T}}$ for all regular triangulations \mathcal{T} of (Q, A) .

Definition 1.6. [BS] Consider the standard simplex with the set of vertices A :

$$\Delta^A = \left\{ (p_{\omega})_{\omega \in A} \in \mathbb{R}^A \mid p_{\omega} \geq 0, \sum_{\omega} p_{\omega} = 1 \right\},$$

so that, by definition of the convex hull, we have a surjective projection

$$\pi : \Delta^A \longrightarrow Q = \text{Conv}(A), \quad (p_{\omega})_{\omega \in A} \mapsto \sum_{\omega} p_{\omega} \cdot \omega.$$

The polytope $\Sigma(A)$ is defined as the set of vector integrals

$$\int_{q \in Q} s(q) \, d\text{Vol} \in \mathbb{R}^A$$

for all continuous sections $s : Q \rightarrow \Delta^A$ of the projection π .

To explain the relation of the two definitions, note that a triangulation \mathcal{T} of (Q, A) defines a “tautological” piecewise linear section $s_{\mathcal{T}} : Q \rightarrow \Delta^A$, and $\phi_{\mathcal{T}} = \int_Q s_{\mathcal{T}}(q) \, d\text{Vol}$.

To each face $F = F_{\mathcal{P}}$ of $\Sigma(A) \subset \mathbb{R}^A$ we can associate its *normal cone* $N_F \subset (\mathbb{R}^A)^*$ consisting of those linear functionals $l : \mathbb{R}^A \rightarrow \mathbb{R}$ which achieve the maximum on F and are constant on F . The cones N_F form a decomposition

(fan) of the space $(\mathbb{R}^A)^*$ called the *normal fan* of $\Sigma(A)$. Let us now identify \mathbb{R}^A with its dual vector space by means of the standard pairing

$$(\phi, \psi) = \sum_{\omega \in A} \phi(\omega)\psi(\omega).$$

Under this identification we have:

Proposition 1.7. *The normal fan of $\Sigma(A)$ is identified with the secondary fan $\mathcal{S}(A)$. In particular:*

(a) *The face lattice of $\Sigma(A)$ is isomorphic to $\mathcal{R}(Q, A)$ in an order-preserving way. We denote by $F_{\mathcal{P}}$ the face corresponding to a subdivision $\mathcal{P} \in \mathcal{R}(Q, A)$.*

(b) *The codimension of $F_{\mathcal{P}}$ is equal to the dimension of the cone $\overline{C}_{\mathcal{P}}$.*

(c) $\dim \Sigma(A) = |A| - d - 1 = \dim(\mathbb{R}^A / \text{Aff}(\mathbb{R}^d))$.

Examples 1.8. (a) A triangulation of (Q, A) can be seen as a polyhedral subdivision $\mathcal{T} = (Q_{\nu}, A_{\nu})$ such that each Q_{ν} is a simplex and A_{ν} consists exactly of the vertices of Q_{ν} . In this case $\Sigma(A_{\nu})$ is a point and so the face $F_{\mathcal{T}}$ is a vertex.

(b) Edges of $\Sigma(A)$ correspond to *flips* (elementary modifications) of triangulations. A flip is based on a *circuit*, a subset $Z \subset A$ which has precisely one, up to a scalar, affine dependency

$$\sum_{\omega \in Z} a_{\omega} \cdot \omega = 0, \quad a_{\omega} \in \mathbb{R}, \quad \sum a_{\omega} = 0.$$

In this case the convex hull $\text{Conv}(Z)$ has precisely two triangulations T_+, T_- with vertices in Z , so $\Sigma(Z)$ is an interval. Each edge $[\phi_{\mathcal{T}}, \phi_{\mathcal{T}'}]$ of $\Sigma(A)$ corresponds to a pair $\mathcal{T}, \mathcal{T}'$ of triangulations of Q which coincide outside $\text{Conv}(Z)$ for some circuit Z , and restrict to T_+, T_- inside $\text{Conv}(Z)$.



Figure 3: Circuits and flips

(c) Codimension 1 faces correspond to *coarse subdivisions* (see [GKZ]). In the dual language used in [GMW] in case $d = 2$ they are called *taut webs*.

By definition, a subdivision \mathcal{P} is coarse, if the reduced normal cone $\overline{C}_{\mathcal{P}}$ has dimension 1, i.e., there exists only one, modulo scaling and adding affine functions, convex \mathcal{P} -piecewise-affine function $f : Q \rightarrow \mathbb{R}$. The following are the most important examples of coarse subdivisions.

(c1) Let $\omega \in A$ be not a vertex of Q . Then $(Q, A - \{\omega\})$ is a coarse subdivision of (Q, A) .

(c2) Let H is a hyperplane in \mathbb{R}^d subdividing Q into two subpolytopes Q_1, Q_2 . Suppose that the vertices of both these polytopes lie in A . Put $A_\nu = A \cap Q_\nu$. Then (Q_1, A_1) and (Q_2, A_2) form a coarse subdivision of (Q, A) .

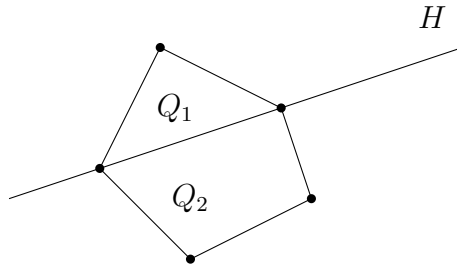


Figure 4: A 2-part coarse subdivision.

The importance of $\Sigma(A)$ for our purposes comes from the following.

Proposition 1.9 (Factorization property). *The face $F_{\mathcal{P}}$ corresponding to a regular polyhedral subdivision $\mathcal{P} = (Q_\nu, A_\nu)$, has the form $F_{\mathcal{P}} = \prod_{\nu} \Sigma(A_\nu)$, i.e., it is itself a product of several secondary polytopes.*

Remark 1.10. Note that if we drop the assumption of A being in general position, then $F_{\mathcal{P}}$ will be not the full product but the fiber product of the $\Sigma(A_\nu)$ over the secondary polytopes of intermediate faces. This one of the complications we wanted to avoid in this paper.

For future convenience, we introduce the following terminology.

Definition 1.11. A marked subpolytope $(Q', A') \subset (Q, A)$ is called *geometric*, if $A' = A \cap Q'$. A polyhedral subdivision $\mathcal{P} = \{(Q'_\nu, A'_\nu)\}$ of (Q, A) is called geometric, if each (Q'_ν, A'_ν) is a geometric marked subpolytope. A face F of $\Sigma(A)$ is called geometric, if it corresponds to a geometric subdivision.

We will use the term *subpolytope* of (Q, A) for a polytope $Q' \subset Q$ with vertices from A . Notation: $Q' \subset (Q, A)$. Such subpolytopes are in bijection with geometric marked subpolytopes $(Q', A \cap Q')$.

The following property will be important for the construction of §3 to make sense. Its proof is obvious and is left to the reader.

Proposition 1.12. *Geometric faces form a right ideal in the partially ordered set of all faces of $\Sigma(A)$. That is, if $F \subset F'$ and F is geometric, then F' is geometric. \square*

We introduce the *geometric part* of the secondary polytope to be the union of the interiors of geometric faces (of all dimensions):

$$(1.13) \quad \Sigma^{\text{geom}}(A) = \bigcup_{\substack{F \subset \Sigma(A) \\ \text{geometric}}} \text{Int}(F).$$

Proposition 1.12 can be reformulated by saying that $\Sigma^{\text{geom}}(A)$ is an open subset of $\Sigma(A)$.

2 A commutative dg-algebra from the chain complex of $\Sigma(A)$.

The factorization property of $\Sigma(A)$ implies that its chain complex looks like a (part of) a multiplicative differential in a dg-algebra.

More precisely, let \mathbf{k} be a field of characteristic 0. For any convex polytope P we denote by $C_\bullet(P)$ its cellular chain complex with coefficients in \mathbf{k} graded so that i -chains are positioned in degree $(-i)$ and the differential raises the degree by $+1$. Thus, as a graded vector space

$$C_\bullet(P) = \bigoplus_{F \subset P} \text{or}(F)[\dim(F)],$$

the direct sum over all faces. Here

$$\text{or}(F) = H_c^{\dim(F)}(\text{Interior of } F, \mathbf{k})$$

is the 1-dimensional orientation space of F .

We apply this to $P = \Sigma(A)$ and form the graded vector space

$$(2.1) \quad V = \bigoplus_{(Q', A') \subset (Q, A)} V_{A'}, \quad V_{A'} = \text{or}(\Sigma(A'))[\dim \Sigma(A')],$$

the direct sum over all marked subpolytopes of full dimension d . Here, the shift means that $\text{or}(\Sigma(A'))$ is positioned in degree $-\dim \Sigma(A')$. Note that if $\mathbf{k} = \mathbb{R}$, then

$$\text{or}(\Sigma(A')) = \Lambda^{\max}(\mathbb{R}^{A'} / \text{Aff}(\mathbb{R}^d)).$$

Now, take the symmetric algebra $S^\bullet(V)$ with the graded commutative product denoted \odot . Each regular polyhedral subdivision $\mathcal{P}'' = \{(Q''_\nu, A''_\nu)\}$ of each marked subpolytope (Q', A') of (Q, A) produces a 1-dimensional subspace

$$(2.2) \quad V_{\mathcal{P}''} = \bigodot_{\nu} V_{A''_\nu} \subset S^\bullet(V).$$

The chain differentials in all the $C_\bullet(\Sigma(A'))$ are compatible with each other and give a differential d of degree $+1$

$$(2.3) \quad d : S^\bullet(V) \longrightarrow S^\bullet(V),$$

satisfying the Leibniz rule.

More precisely, let $\mathcal{P}'' = \{(P''_\nu, Q''_\nu)\}$ be a coarse polyhedral subdivision of (Q', A') , $A' \subset A$. Using the factorization property (Proposition 1.9), we see that the top degree part of the chain differential in $C_\bullet(\Sigma(A'))$ gives a map

$$d_{\mathcal{P}''} : V_{A'} \longrightarrow V_{\mathcal{P}''} \subset S^\bullet(V).$$

We define d to be given, on generators, by

$$d|_{V_{A'}} = \sum_{\substack{\mathcal{P}'' \text{ coarse} \\ \text{subdiv. of } (Q', A')}} d_{\mathcal{P}''},$$

and then extend it to the whole of $S^\bullet(V)$ by the Leibniz rule.

Proposition 2.4. *The differential d thus defined, satisfies $d^2 = 0$ and so makes $S^\bullet(V)$ into a commutative dg-algebra.*

Proof: In virtue of the Leibniz rule, it is enough to check that $d^2 = 0$ on generators. To see this, note that on $V_{A'}$ the picture for d^2 is embedded, again by Proposition 1.9, into the picture for d^2 in $C_\bullet(\Sigma(A'))$. But for any polytope P , the differential in the chain complex $C_\bullet(P)$ satisfies $d^2 = 0$. \square

Remark 2.5. We take the *symmetric* algebra of V because, for a subdivision $\mathcal{P}'' = \{(Q''_\nu, A''_\nu)\}$ of (Q', A') , different marked polytopes (Q''_ν, A''_ν) have no apparent order. But this can be refined depending on the dimension d , as we will see later.

3 The L_∞ -algebra.

If \mathfrak{h} is a finite-dimensional dg-Lie algebra over \mathbf{k} , then its Chevalley-Eilenberg cochain complex is a commutative dg-algebra

$$C_{\text{Lie}}^\bullet(\mathfrak{h}) = S^\bullet(\mathfrak{h}^*[-1]).$$

For any graded vector space V we denote by $S_+^\bullet(V) \subset S^\bullet(V)$ the maximal ideal $\bigoplus_{n \geq 1} V^{\otimes n}$.

As well known, a general algebra differential d on $S_+^\bullet(\mathfrak{h}^*[-1])$ (i.e. an algebra derivation satisfying $d^2 = 0$), makes \mathfrak{h} into an L_∞ -algebra, see, e.g., [KoSo1], Ch.3, Section 2.3. Equivalently we can say that L_∞ -algebra on \mathfrak{h} is given by the algebra differential of $S^\bullet(\mathfrak{h}^*[-1])$ which preserves $S_+^\bullet(\mathfrak{h}^*[-1])$.

Thus, in the situation of §2, we get an L_∞ -structure on the vector space

$$\dot{\mathfrak{g}} = \dot{\mathfrak{g}}_A = V^*[-1] = \bigoplus_{(Q', A') \subset (Q, A)} E_{A'},$$

$$E_{A'} = V_{A'}^*[-1] = \text{or}(\Sigma(A'))^*[-1 - \dim \Sigma(A')].$$

In other words, the 1-dimensional \mathbf{k} -vector space $\text{or}(\Sigma(A))^*$ (which we can canonically identify with its dual $\text{or}(\Sigma(A))$), is now put in the positive degree $1 + \dim \Sigma(A')$.

The L_∞ -algebra $\dot{\mathfrak{g}}$ is, while natural, too big for our purposes. Indeed, a subpolytope $Q' \subset (Q, A)$ with vertices in A can give rise to many summands $E_{A'} \subset \dot{\mathfrak{g}}$ corresponding to the choice of A' sandwiched between $\text{Vert}(Q')$ and $A \cap Q'$. The dot in the notation $\dot{\mathfrak{g}}$ is supposed to symbolize this freedom in choosing the set of points A' . Taking only the “geometric summand corresponding to $A' = A \cap Q'$ ”, we define the subspace

$$\mathfrak{g} = \mathfrak{g}_A = \bigoplus_{(Q', A') \text{ geom.}} E_{A'} = \bigoplus_{Q' \subset (Q, A)} E_{A \cap Q}; \subset \dot{\mathfrak{g}}.$$

Remark 3.1. We should warn the reader that in the Sections 3-8 we discuss L_∞ -algebras which are toy models of the “realistic” L_∞ -algebras appearing in the “nature” (in particular, in the study of Fukaya-Seidel categories). For the latter situation one has to introduce a “coefficient system” on the secondary polytope. This will be done in Section 9. Nevertheless the case of “trivial coefficient system” discussed in Sections 3-8 has most of the features of the general case, so we decided to keep it for pedagogical reasons.

Proposition 3.2. \mathfrak{g}_A is an L_∞ -subalgebra in $\dot{\mathfrak{g}}_A$, with trivial differential.

The L_∞ -algebra \mathfrak{g}_A (together with its various generalizations) will be the primary object of study in this paper. We will call it the *geometric L_∞ -algebra* associated to A .

Proof of Proposition 3.2: We denote by

$$\lambda_n : \dot{\mathfrak{g}}^{\otimes n} \longrightarrow \dot{\mathfrak{g}}, \quad n = 1, 2, \dots,$$

the components of the L_∞ -structure. Thus $\lambda_1 = d$ is the differential, $\lambda_2(x \otimes y) = [x, y]$ is the bracket, etc.

It follows from our construction that λ_n corresponds to coarse subdivisions of marked subpolytopes of (Q, A) into precisely n marked subpolytopes. More precisely, choose a generator (canonical up to sign) $e_{A'} \in \text{or}(\Sigma(A'))$ for each $A' \subset A$, $|A'| = d + 1$. Thus, the $e_{A'}$ form a basis in \mathfrak{g} . Then each coarse subdivision $\mathcal{P}'' = \{(Q''_\nu, A''_\nu)\}_{\nu=1}^n$ of each (Q', A') contributes a matrix element

$$(3.3) \quad \langle e_{A'_1} \otimes \cdots \otimes e_{A'_n} \mid \lambda_n \mid e_{A'} \rangle = \pm 1,$$

and all non-zero matrix elements of λ_n are obtained in this way. Now, if each (Q''_ν, A''_ν) is geometric, i.e., $A''_\nu = A \cap Q''_\nu$, the $A' = A \cap Q'$ as well, i.e., (Q', A') is geometric, cf. Proposition 1.12. This means that $\lambda_n(\mathfrak{g}^{\otimes n}) \subset \mathfrak{g}$, i.e., that \mathfrak{g} is an L_∞ -subalgebra. The fact that the differential in \mathfrak{g} vanishes, follows from Example 3.5(a) below. \square

Example 3.4. Recall that we have assumed that A is in affinely general position, i.e. each $(d + 1)$ -element subset generates a d -simplex. Hence the low degree part of the L_∞ -algebra \mathfrak{g} looks as follows. We have $\mathfrak{g}^{\leq 0} = 0$, while \mathfrak{g}^1 is spanned by d -dimensional simplices, i.e., by marked subpolytopes (Q', A') with $|A'| = d + 1$. The degree 2 part \mathfrak{g}^2 is spanned by circuits $(|A'| = d + 2)$ and so on.

Examples 3.5. (a) Differential in $\dot{\mathfrak{g}}$. It corresponds to Example 1.8(c1) of coarse subdivisions. That is, given $A' \subset A$ and $\omega \in A'$ which is not a vertex of $Q' = \text{Conv}(A')$, we have, specializing (3.3), that

$$\langle e_{A' - \{\omega\}} \mid d \mid e_{A'} \rangle = \pm 1.$$

Writing B for $A' - \{\omega\}$ (which can be arbitrary) and collecting the matrix elements (i.e., adding, not subtracting one point in all admissible ways), we have, cf. Fig. 5:

$$d(e_B) = \sum_{\omega \in (A \cap Q') - B} \pm e_{B \cup \{\omega\}}, \quad Q' = \text{Conv}(B).$$

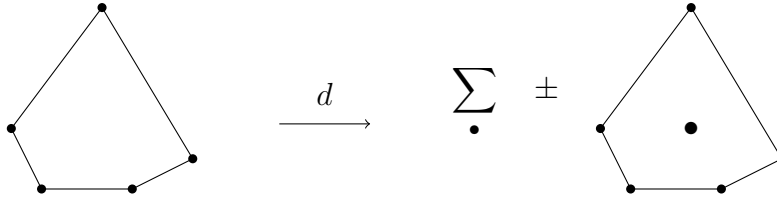


Figure 5: Differential in \mathfrak{g} .

In other words, fixing a subpolytope $Q' \subset (Q, A)$, all e_B with $\text{Conv}(B) = Q'$, span a subcomplex in (\mathfrak{g}, d) . This subcomplex is isomorphic to the augmented cochain complex of the simplex Δ^I , whose set of vertices is

$$I = (A \cap Q') - \text{Vert}(Q').$$

In particular, it is exact, if $I \neq \emptyset$, i.e., if Q' contains points of A other than its vertices. This exactness is another reason why \mathfrak{g} is more important for us than \mathfrak{g} .

Note that if $B = A \cap Q'$, then $d(e_B) = 0$ (there are no more points to add), so the differential in \mathfrak{g} vanishes.

(b) The binary bracket in \mathfrak{g} corresponds to Example 1.8(c2) of coarse subdivisions. That is, if (Q', A') is subdivided by a hyperplane H into (Q''_1, A''_1) and (Q''_2, A''_2) , then

$$[e_{A''_1}, e_{A''_2}] = \pm e'_A$$

(c) Consider the particular case $d = 2$ and assume that $A \subset \mathbb{R}^2$ is in *convex position*, i.e., each element of A is a vertex of $Q = \text{Conv}(A)$. Then $\mathfrak{g} = \mathfrak{g}$ and only the binary bracket is present, so \mathfrak{g} is a graded Lie algebra in a more familiar sense (no differential, no higher λ_n). As an illustration,

consider the further particular case when A consists of 4 points, forming the vertices of a convex 4-gon. Then \mathfrak{g}^1 has dimension 4, with the basis vectors corresponding to the 4 triangles a, b, c, d in Fig.6. The space \mathfrak{g}^2 is 1-dimensional, with the basis vector corresponding to the 4-gon itself, while $\mathfrak{g}^{\geq 3} = 0$. Thus \mathfrak{g} has five basis vectors: e_a, e_b, e_c, e_d of degree 1 and e_A of degree 2, with the only non-zero brackets being

$$[e_a, e_b] = [e_c, e_d] = e_A.$$

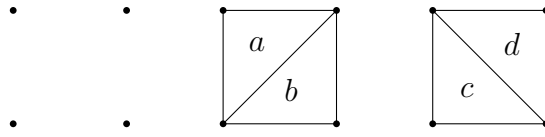


Figure 6: Four points in convex position

(d) Take $A \subset \mathbb{R}^2$ to be a 4-element circuit consisting of the 3 vertices of a triangle Q and one more point ω' inside Q . Marked subpolytopes of (Q, A) include 3 small triangles a, b, c in Fig. 7, one large triangle $f = (Q, A - \{\omega'\})$, and (Q, A) itself. In this case \mathfrak{g} has two non-zero components: \mathfrak{g}^1 , with basis e_a, e_b, e_c, e_f and \mathfrak{g}^2 with basis e_A . The only non-trivial parts of the L_∞ -structure are the differential and the ternary bracket whose nontrivial matrix elements are given by:

$$d(e_f) = e_A, \quad \lambda_3(e_a, e_b, e_c) = e_A,$$

so the cohomology (w.r.t. d) of \mathfrak{g} has all the multiplications λ_n being 0. On the other hand, \mathfrak{g} is spanned by e_a, e_b, e_c, e_A and has $\lambda_3 \neq 0$.

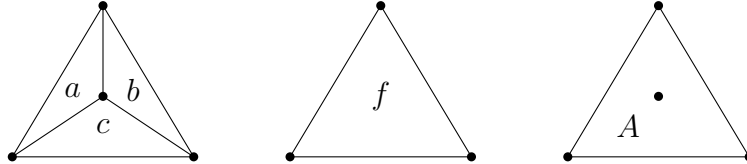


Figure 7: Four points in non-convex position.

Recall [Get] that an L_∞ -algebra \mathcal{L} is called *nilpotent*, if there exists $r_0 > 0$ such that all r -ary iterated superpositions of the λ_n are identically zero as maps $\mathcal{L}^{\otimes r} \rightarrow \mathcal{L}$ for $r > r_0$.

Proposition 3.6. *The L_∞ -algebra \mathfrak{g}^\bullet (and therefore \mathfrak{g}) is nilpotent.*

Proof: Since the matrix elements of λ_n corresponds to *coarse* subdivisions of marked subpolytopes $(Q', A') \subset (Q, A)$ into n marked subpolytopes, the matrix elements of any r -ary iterated superposition of the λ_n correspond to some (not necessarily coarse) subdivisions into r marked subpolytopes and therefore vanish for $r \geq |A|$. \square

4 Maurer-Cartan elements in \mathfrak{g} .

Let \mathcal{L} be a nilpotent L_∞ -algebra over \mathbf{k} . Recall that a *Maurer-Cartan element* in \mathcal{L} is an element $\gamma \in \mathcal{L}^1$ such that

$$(4.1) \quad \sum_{n=0}^{\infty} \frac{1}{n!} \lambda_n(\gamma, \dots, \gamma) = 0 \in \mathcal{L}^2.$$

Maurer-Cartan elements form a subset $\text{MC}(\mathcal{L})$ in \mathcal{L}^1 . If $\dim \mathcal{L}^1 < \infty$, we can consider \mathcal{L}^1 as an affine space over \mathbf{k} and view $\text{MC}(\mathcal{L})$ as an affine subscheme in \mathcal{L}^1 .

If \mathcal{L} is a general L_∞ -algebra, one can define the *formal scheme of Maurer-Cartan elements* $\widehat{\text{MC}}(\mathcal{L})$ which is the functor on local Artin \mathbf{k} -algebras Λ defined by

$$\Lambda \mapsto \text{MC}(\mathcal{L} \otimes_{\mathbf{k}} \mathfrak{m}),$$

where \mathfrak{m} is the maximal ideal of Λ .

Such a functor represents a geometric object called a *formal pointed dg-manifold* in [KoSo1], Ch.3.

We are interested in Maurer-Cartan elements in the finite-dimensional nilpotent L_∞ -algebra $\mathfrak{g} = \mathfrak{g}_A$. In this section we assume that A is in general position. As we saw, \mathfrak{g}^1 is spanned by basis vectors e_σ corresponding to all possible marked d -simplices $\sigma = (\text{Conv}(A'), A')$ with $A' \subset A$, $|A'| = d + 1$. Such a marked simplex is nothing but a straight d -simplex on vertices from A . Thus we can view

$$\gamma = \sum \gamma_\sigma \cdot e_\sigma, \quad \gamma_\sigma \in \mathbf{k}$$

as a “ d -cochain”, i.e., a datum of numbers $\gamma_\sigma \in \mathbf{k}$ attached to all marked simplices σ .

Now, \mathfrak{g}^2 is spanned by basis vectors e_Z corresponding to all possible circuits $Z \subset A$, $|Z| = d + 2$. The coefficient of the LHS of (4.1) at e_Z is then equal to the sum of two terms corresponding to the two triangulations T_+, T_- of $\text{Conv}(Z)$, as in Example 1.8(b). More precisely, this coefficient is equal to

$$\prod_{\sigma \in T_+} \gamma_\sigma - \prod_{\sigma \in T_-} \gamma_\sigma.$$

Therefore we obtain:

Proposition 4.2. γ is a Maurer-Cartan element in \mathfrak{g} , if and only if it is a “cocycle” in the multiplicative sense: for any circuit Z with triangulations T_+, T_- of $\text{Conv}(Z)$ we have

$$\prod_{\sigma \in T_+} \gamma_\sigma = \prod_{\sigma \in T_-} \gamma_\sigma. \quad \square$$

This means that for any marked subpolytope (Q', A') of (Q, A) we have an element $\gamma_{Q', A'} \in \mathbf{k}$ defined in terms of any chosen regular triangulation \mathcal{T}' of (Q', A') :

$$\gamma_{Q', A'} = \prod_{\sigma \in \mathcal{T}'} \gamma_\sigma,$$

with the RHS being independent of the choice of \mathcal{T}' .

Example 4.3. Let $\mathbf{k} = \mathbb{C}$ and let Ω be any smooth d -form on Q . Then putting

$$\gamma_\sigma = e^{\beta_\sigma}, \quad \beta_\sigma = \int_\sigma \Omega,$$

we get a Maurer-Cartan element in \mathfrak{g} .

One way of interpreting Proposition 4.2 is by saying that the subscheme $\text{MC}(\mathfrak{g})$ is given inside the affine space \mathfrak{g}^1 by a system of *binomial equations*: each equation had the form of equality of exactly two monomials. This means that the scheme $\text{MC}(\mathfrak{g})$ is in fact a toric variety.

More precisely, let $\mathfrak{S}_d(A)$ be the set of all d -simplices on vertices from A . Thus $\mathfrak{S}_d(A)$ is a subset in the set $\binom{A}{d+1}$ of all $(d+1)$ -element subsets in A , and $\mathfrak{S}_d(A) = \binom{A}{d+1}$ when A is in affinely general position. Let $\mathbb{Z}^{\mathfrak{S}_d(Q, A)}$ be the free abelian group of *combinatorial d -chains* in (Q, A) , i.e., of integer linear combinations of d -simplices. We write an element $\beta \in \mathbb{Z}^{\mathfrak{S}_d(Q, A)}$ as a system $\beta = (\beta_\sigma)$ where $\sigma \in \mathfrak{S}_d(A)$ is a d -simplex, and $\beta_\sigma \in \mathbb{Z}$. Let $\mathfrak{Z}_d(Q, A) \subset \mathbb{Z}^{\mathfrak{S}_d(Q, A)}$ be the sublattice of *d -cocycles*, i.e., of β as above such that for each circuit Z as in Proposition 4.2, we have

$$\sum_{\sigma \in T_+} \beta_\sigma = \sum_{\sigma \in T_-} \beta_\sigma.$$

Let $\Gamma_A = \mathfrak{Z}_d(Q, A) \cap \mathbb{Z}_+^{\mathfrak{S}_d(Q, A)}$. This is a finitely generated abelian semigroup. The monomial equations for $\text{MC}(\mathfrak{g})$ translate into the following.

Corollary 4.4. *The scheme $\text{MC}(\mathfrak{g})$ is isomorphic to $\text{Spec } \mathbf{k}[\Gamma_A]$, the spectrum of the semigroup algebra of Γ_A . \square*

5 1-dimensional case: refinement to an A_∞ -algebra.

In this section we consider the simplest case $d = 1$. So we assume that $A = \{\omega_1 < \dots < \omega_r\} \subset \mathbb{R}$. Then $Q = \text{Conv}(A)$ is the interval $[\omega_1, \omega_r]$. Let us form the vector space V as in (2.1).

If (Q', A') is a marked subpolytope of (Q, A) and $\mathcal{P}'' = \{(Q''_\nu, A''_\nu)\}$ is a polyhedral subdivision of (Q', A') , then each Q''_ν is a sub-interval in Q' and so *there is a natural order* on the set of the intervals Q''_ν , induced by the order on \mathbb{R} (i.e., geometrically, from left to right). This means that we can lift the differential d from $S^\bullet(V)$ (free commutative algebra generated by V) to a differential in $T^\bullet(V)$, the tensor algebra (free associative algebra) generated by V . More precisely, we realize the vector space $V_{\mathcal{P}''}$ from (2.2) as a subspace in $T^\bullet(V)$:

$$V_{\mathcal{P}''} = \bigotimes_{\nu} V_{A''_\nu} \subset T^\bullet(V)$$

(embedding given by the tensor multiplication in the order from left to right). We then define the action of d on the generators of $T^\bullet(V)$ by making it act on a summand $V_{A'} \subset V$ as the top degree part of the chain differential in $C_\bullet(\Sigma(A'))$, now considered as a map $\sum_{\mathcal{P}''} d_{\mathcal{P}''}$, where \mathcal{P}'' runs over all coarse subdivisions of (Q', A') and

$$d_{\mathcal{P}''} : V_{A'} \longrightarrow V_{\mathcal{P}''} \subset T^\bullet(V)$$

is induced by the chain differential in $C_\bullet(\Sigma(A'))$. Then we extend d to the whole $T^\bullet(V)$ by the Leibniz rule. Similarly to Proposition 2.4, we then get:

Proposition 5.1. *The differential d in $T^\bullet(V)$ satisfies $d^2 = 0$ and so makes $T^\bullet(V)$ into an associative dg-algebra. The canonical commutativization homomorphism $(T^\bullet(V), d) \rightarrow (S^\bullet(V), d)$ is a morphism of associative dg-algebras. \square*

Clearly d preserves the ideal $T_+^\bullet(V) := \bigoplus_{n \geq 1} V^{\otimes n} \subset T^\bullet(V)$.

This means that the L_∞ -algebra \mathfrak{g} lifts to an A_∞ algebra with the same underlying vector space by the A_∞ -analog of the formula

$$[x, y] = xy - yx.$$

We denote this A_∞ -algebra by \dot{R} . As before, we have a similar lifting of the subalgebra $\dot{\mathfrak{g}}$. We denote this lifting by R and it will be more fundamental for us than \dot{R} .

Thus R has basis e_{ij} , $1 \leq i < j \leq r$, corresponding to the geometric marked subpolytopes

$$([\omega_i, \omega_j], \{\omega_i, \omega_{i+1}, \dots, \omega_j\}).$$

The degree of e_{ij} is $j - i$, and the binary multiplication is given by

$$e_{ij} \cdot e_{jk} = e_{ik}, \quad i < j < k,$$

and all other binary products as well as other A_∞ -data vanish. In other words, R is the associative algebra without unit, isomorphic to $T_n(\mathbf{k})$, the algebra of strictly upper triangular r by r matrices with zeroes on and below the diagonal. This triangular structure is the simplest instance of the relation between our formalism and deformations of semi-orthogonal collections, see below.

6 Relative setting: one point “at infinity”.

Let (Q, A) be a marked polytope in \mathbb{R}^d , as above. Let us add to A one more point $\infty \in \mathbb{R}^d$, forming $\tilde{A} = A \cup \{\infty\}$ and $\tilde{Q} = \text{Conv}(\tilde{A})$. We assume that ∞ is a new vertex of \tilde{Q} , i.e., that it lies “far outside of A ”. We also assume that the set \tilde{A} is in general position. Let $\tilde{\mathfrak{g}} = \mathfrak{g}_{\tilde{A}}$ be the geometric L_∞ -algebra corresponding to \tilde{A} .

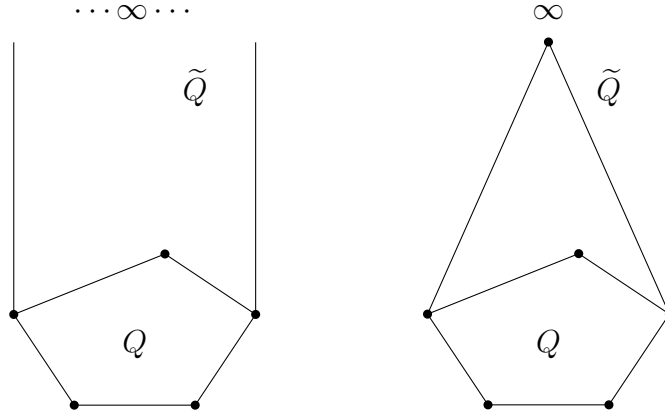


Figure 8: ∞ as a point at the projective infinity vs. a finite point far away.

Remark 6.1. We can, if we want, imagine ∞ to be an actual “infinite” point of the projective completion $\mathbb{RP}^d = \mathbb{R}^d \cup \mathbb{RP}^{d-1}$, say the point of \mathbb{RP}^{d-1} with homogeneous coordinates $[0 : 0 : \cdots : 1]$. Note that the concepts of hyperplanes, triangulations, convexity etc. are invariants under projective transformations of \mathbb{RP}^d . So there is no combinatorial difference between a bounded polytope \tilde{Q} on the right and the unbounded polyhedron \tilde{Q} on the left of Fig. 8.

A subpolytope $Q' \subset (\tilde{Q}, \tilde{A})$ will be called *finite*, if $\text{Vert}(Q') \subset A$, and *infinite*, otherwise, i.e., if $\infty \in Q'$. The L_∞ -algebra $\tilde{\mathfrak{g}}$ splits, as a vector space, into a direct sum

$$\tilde{\mathfrak{g}} = \mathfrak{g} \oplus \mathfrak{g}_\infty,$$

where $\mathfrak{g} = \mathfrak{g}_A$ is spanned by the summands

$$E_{A'} = \text{or}(\Sigma(Q' \cap A))^*[-1 - \dim \Sigma(Q' \cap A)]$$

for finite Q' , and \mathfrak{g}_∞ is spanned by similar summands for infinite Q' .

Proposition 6.2. *Both \mathfrak{g} and \mathfrak{g}_∞ are L_∞ -subalgebras of $\tilde{\mathfrak{g}}$, and, moreover, \mathfrak{g}_∞ is an L_∞ -ideal. In other words, $\tilde{\mathfrak{g}}$ is the semi-direct product $\tilde{\mathfrak{g}} = \mathfrak{g} \ltimes \mathfrak{g}_\infty$.*

Proof: The L_∞ -brackets are given by combining subpolytopes, say, Q''_1, \dots, Q''_n , together into a coarse subdivision. If neither of the Q''_i contains ∞ , then the combined marked polytope Q' does not contain it either. This means that \mathfrak{g} is an L_∞ -subalgebra (it is nothing but \mathfrak{g}_A , the geometric L_∞ -algebra associated to A alone). If at least one of the Q''_i does contain ∞ , then so does Q' . This means that \mathfrak{g}_∞ is a L_∞ -ideal. \square

Remarks 6.3. (a) Let us choose a Euclidean subspace $\mathbb{R}^{d-1} \subset \mathbb{R}^d$ not passing through ∞ and let

$$p : \mathbb{R}^d - \{\infty\} \longrightarrow \mathbb{R}^{d-1}$$

be the projection obtained by drawing straight lines through ∞ and intersecting them with \mathbb{R}^{d-1} . In the “projective infinity” picture on the left of Fig. 8, we can think of $p : \mathbb{R}^d \rightarrow \mathbb{R}^{d-1}$ as being the projection forgetting the last coordinate.

We then have the point configuration $\bar{A} = p(A)$ in \mathbb{R}^{d-1} , and by our assumption, $\bar{Q} = \text{Conv}(\bar{A})$ coincides with the image $p(Q)$. It is instructive to compare the L_∞ -algebra \mathfrak{g}_∞ with $\mathfrak{g}_{\bar{A}}$, the L_∞ -algebra associated to $\bar{A} \subset \mathbb{R}^{d-1}$.

More precisely, each infinite subpolytope $Q' \subset (\tilde{Q}, \tilde{A})$ gives a subpolytope $p(Q') \subset (\bar{Q}, \bar{A})$ via the projection p . However, not every subpolytope of (\bar{Q}, \bar{A}) is obtained in this way. Similarly, each coarse subdivision of Q' gives a coarse subdivision of $p(Q')$ but not the other way around. Nevertheless, the L_∞ -algebra \mathfrak{g}_∞ has, in many respects, $(d-1)$ -dimensional nature.

(b) The two point configurations A, \bar{A} associated to \tilde{A} and an element $\infty \in \tilde{A}$ are precisely the two *minors* of \tilde{A} studied by Billera, Gelfand and Sturmfels [BGS]. More precisely, A is the *minor by deletion* and \bar{A} is the *minor by contraction*, in the terminology of [BGS]. Among other things, it was proved in [BGS] that $\Sigma(\bar{A})$ is a face of $\Sigma(\tilde{A})$, and $\Sigma(A)$ is a Minkowski summand of $\Sigma(\tilde{A})$.

Remark 6.4. If we assume that $\tilde{\mathfrak{g}}$ is an ordinary dg-Lie algebra, i.e. if $\lambda_{\geq 3} = 0$, then the Lie algebra structure on $\tilde{\mathfrak{g}}$ gives a morphism of dg-Lie algebras

$$\mathfrak{g} \longrightarrow \text{Der}(\mathfrak{g}_\infty), \quad x \mapsto \text{ad}(x) : \mathfrak{g}_\infty \rightarrow \mathfrak{g}_\infty.$$

In general, what we get is a “derived” analog of such a morphism, as we explain in the next section.

7 Hochschild complexes and derived derivation spaces.

In this section we recall some general principles of deformation theory and of Koszul duality.

Let \mathcal{P} be any dg-operad (over \mathbf{k}), and \mathcal{A} be any \mathcal{P} -algebra (in the category of dg-vector spaces over \mathbf{k}). Automorphisms of \mathcal{A} form a group $\text{Aut}_{\mathcal{P}}(\mathcal{A})$, and derivations of \mathcal{A} (of all degrees) form a dg-Lie algebra $\text{Der}_{\mathcal{P}}(\mathcal{A})$. One can form the derived functor of $\mathcal{A} \mapsto \text{Der}_{\mathcal{P}}(\mathcal{A})$, defining

$$R\text{Der}_{\mathcal{P}}(\mathcal{A}) = \text{Der}_{\mathcal{P}}(\tilde{\mathcal{A}}),$$

where $\tilde{\mathcal{A}} \rightarrow \mathcal{A}$ is a “cofibrant resolution”, i.e., a quasi-isomorphism of \mathcal{P} -algebras with $\tilde{\mathcal{A}}$ being free as a \mathcal{P} -algebra without differential. RHS of this formula can be understood (at least in $\text{char}(\mathbf{k}) = 0$ case) as an L_{∞} -algebra. There is a canonical choice of a cofibrant resolution called the Boardmann-Vogt resolution in [KoSo2]. It was used in *loc. cit.* for the description of general deformation theory of algebras over operads.

Remark 7.1. If $\mathcal{P} \rightarrow \mathcal{Q}$ is a morphism of (dg-)operads, then any \mathcal{Q} -algebra \mathcal{A} can be also considered as a \mathcal{P} -algebra (for example, we have $\mathcal{A}ss \rightarrow \mathcal{C}om$, so a commutative algebra can be considered as a particular case of an associative algebra). In this case $\text{Der}_{\mathcal{P}}(\mathcal{A}) = \text{Der}_{\mathcal{Q}}(\mathcal{A})$, but $R\text{Der}_{\mathcal{P}}(\mathcal{A})$ can be different from $R\text{Der}_{\mathcal{Q}}(\mathcal{A})$.

According to the general philosophy of deformation theory (see e.g. [KoSo1]) the aim of formal deformation theory in the case when $\text{char}(\mathbf{k}) = 0$ is the construction of formal pointed dg-manifold (equivalently, L_{∞} -algebra) which describes the local structure of the moduli space of deformations. More precisely, if \mathcal{L} is such an L_{∞} -algebra, then the commutative topological algebra generated by $\mathcal{L}^*[-1]$ carries a structure of a dg-algebra. Its formal spectrum (as a graded topological algebra) can be thought of as a formal graded scheme, endowed with a (graded) vector field Q of degree +1 such that $[Q, Q] = 0$ (“homological vector field”). The formal scheme of zeros $Z(Q)$ admits a foliation generated by graded vector fields of the form $[Q, \bullet]$ (all schemes are understood as functors on commutative Artin algebras). The “space of leaves” of the above foliation should be thought of as the moduli space of the deformation problem controlled by \mathcal{L} . The reader can look at [KoSo1] for the details and examples and at [KoSo2] for the implementation

of this approach in the case of deformations of algebras over operads. In the more conventional language of stacks one has the following corollary of those considerations.

Proposition 7.2 (Local structure of the moduli stack). *The formal germ of the deformation stack of \mathcal{A} as a \mathcal{P} -algebra is isomorphic to the quotient stack*

$$\widehat{\text{MC}}(R\text{Der}_{\mathcal{P}}(\mathcal{A})) // \exp(R\text{Der}_{\mathcal{P}}^0(\mathcal{A})).$$

Here $\widehat{\text{MC}}(R\text{Der}_{\mathcal{P}}(\mathcal{A}))$ is the formal scheme of Maurer-Cartan elements in the dg-Lie algebra $R\text{Der}_{\mathcal{P}}(\mathcal{A})$, and $\exp(R\text{Der}_{\mathcal{P}}^0(\mathcal{A}))$ is the formal group associated to the Lie algebra $R\text{Der}_{\mathcal{P}}^0(\mathcal{A})$, which naturally acts on the formal scheme $\widehat{\text{MC}}(R\text{Der}_{\mathcal{P}}(\mathcal{A}))$. \square

So we can say that $R\text{Der}_{\mathcal{P}}(\mathcal{A})$ “governs deformations of \mathcal{A} as a \mathcal{P} -algebra”.

Example 7.3. Let $\mathcal{P} = \mathcal{A}ss$, and \mathcal{A} be an associative algebra. Then $R\text{Der}(\mathcal{A}) = R\text{Der}_{\mathcal{A}ss}(\mathcal{A})$ is the shifted and truncated Hochschild cochain complex of \mathcal{A} :

$$C^{\geq 1}(\mathcal{A}, \mathcal{A})[1] := C_{\mathcal{A}ss}^{\geq 1}(\mathcal{A}, \mathcal{A})[1] = \left\{ \text{Hom}_k(A, A) \xrightarrow{\delta_0} \text{Hom}_k(A^{\otimes 2}, A) \xrightarrow{\delta_1} \text{Hom}_k(A^{\otimes 3}, A) \rightarrow \dots \right\}.$$

Thus $\text{Ker}(\delta_0) = \text{Der}(\mathcal{A})$ is the usual space of derivations. The Lie algebra structure on $R\text{Der}(\mathcal{A})$ is given by the “brace formula”. That is, for $p \in \text{Hom}(A^{\otimes m}, A)$ and $q \in \text{Hom}(A^{\otimes n}, A)$ and $i = 1, \dots, m$ we define the i th brace of p and q by $p \circ_i q \in \text{Hom}(A^{\otimes(m+n-1)}, A)$ by

$$\begin{aligned} (p \circ_i q)(a_1, \dots, a_{m+n-1}) &= \\ &= p(a_1, \dots, a_{i-1}, q(a_i, a_{i+1}, \dots, a_{i+n-1}), a_{i+n}, \dots, a_{m+n-1}). \end{aligned}$$

Then

$$[p, q] = \sum_{i=1}^m \pm p \circ_i q - \sum_{j=1}^n \pm q \circ_j p.$$

Example 7.4. Let $\mathcal{P} = \mathcal{L}ie$ and $\mathcal{A} = \mathcal{L}$ be a Lie algebra. Then $R\text{Der}_{\mathcal{L}ie}(\mathcal{L}) =$ is the shifted and truncated Chevalley-Eilenberg cochain complex of \mathcal{L} with coefficients in the adjoint representation:

$$C_{\mathcal{L}ie}^{\geq 1}(\mathcal{L}, \mathcal{L})[1] = \left\{ \mathcal{L}^* \otimes \mathcal{L} \xrightarrow{\delta_0} \Lambda^2 \mathcal{L}^* \otimes \mathcal{L} \xrightarrow{\delta_0} \Lambda^3 \mathcal{L}^* \otimes \mathcal{L} \rightarrow \dots \right\}$$

General principles of Koszul duality: Let \mathcal{P} be a quadratic Koszul operad with Koszul dual operad $\mathcal{P}^!$ (see e.g. [GiKa] about basics on Koszul duality for operads). A *weak \mathcal{P} -algebra structure* (also called a *\mathcal{P}_∞ -algebra structure*) on a graded vector space \mathcal{A} is the same as a $\mathcal{P}^!$ -algebra differential d on the free $\mathcal{P}^!$ -algebra $F_{\mathcal{P}^!}(\mathcal{A}^*[-1])$, satisfying $d^2 = 0$.

We remark that we define the free $\mathcal{P}^!$ -algebra $F_{\mathcal{P}^!}(\mathcal{A}^*[-1])$ in such a way that it does not have 0-ary operations.

In particular:

(a) An L_∞ -structure on a graded vector space \mathcal{L} is a multiplicative differential on the (completed) symmetric algebra $S_+^\bullet(\mathcal{L}^*[-1])$.

(b) An A_∞ -structure on a graded vector space R is a multiplicative differential on the (completed) tensor algebra $T_+^\bullet(R^*[-1])$.

Although technically one should consider completed tensor products (since corresponding structures are defined in dual terms of the differential cofree coalgebras), we will often skip the word “completed”, since our examples are nilpotent in the sense that only finitely many “higher” operations are non-trivial.

Most important for us will be the following principle.

Proposition 7.5. *A datum consisting of:*

- (1) A \mathcal{P}_∞ -algebra \mathcal{A} .
- (2) An L_∞ -algebra \mathcal{L} .
- (3) An L_∞ -morphism $\alpha : \mathcal{L} \rightarrow R\text{Der}_{\mathcal{P}_\infty}(\mathcal{A})$

is the same as a datum consisting of:

- (i) A differential $d_{\mathcal{L}}$ in the (completed) algebra $S^\bullet(\mathcal{L}^*[-1])$ which preserves the maximal ideal $S_+^\bullet(\mathcal{L}^*[-1])$.
- (ii) A differential d in the (completed) graded space

$$S^\bullet(\mathcal{L}^*[-1]) \otimes_{\mathbf{k}} F_{\mathcal{P}^!}(\mathcal{A}^*[-1])$$

which is a $\mathcal{P}^!$ -algebra derivation, satisfies $d^2 = 0$ as well as the following Leibniz formula:

$$d(a \otimes x) = d_{\mathcal{L}^*[-1]}(a) \otimes x + (-1)^{|a|} a \otimes d(1 \otimes x), \quad a \in S^\bullet(\mathcal{L}^*[-1]), \quad x \in F_{\mathcal{P}^!}(\mathcal{A}^*[-1]).$$

Proof: We explain a more conceptual argument and then spell out additional algebraic details in the examples we will be interested in. Suppose data (i)-(ii) are given. Then (i) implies that \mathcal{L} is an L_∞ -algebra, thus giving the datum (2). Further, considering the restriction $d|_{\mathcal{A}^*[-1]}$ and projecting the image on $1 \otimes F_{\mathcal{P}^!}(\mathcal{A}^*[-1])$, we get a $\mathcal{P}^!$ -algebra differential in $F_{\mathcal{P}^!}(\mathcal{A}^*[-1])$, i.e., we get a \mathcal{P}_∞ -algebra structure on \mathcal{A} which we will call the “initial” structure. This gives the datum (1). Now, by general principles of deformation theory, $\text{Spec } S^\bullet(\mathcal{L}^*[-1])$ is a formal commutative dg-manifold (functor on local artinian commutative dg-algebras) whose underlying ordinary (non-dg) formal commutative manifold is $\widehat{\text{MC}}(\mathcal{L})$, the formal scheme of Maurer-Cartan elements in \mathcal{L} . So we denote

$$\widehat{\text{RMC}}(\mathcal{L}) = \text{Spec } S^\bullet(\mathcal{L}^*[-1]).$$

Thus our differential d in the tensor product is the same as a family of \mathcal{P}_∞ -structures on \mathcal{A} parametrized by $\widehat{\text{RMC}}(\mathcal{L})$ which gives the datum (3) by Proposition 7.2. The argument in the opposite direction is similar. \square

Example 7.6 (Particular case $\mathcal{P} = \mathcal{A}_{SS}$). A datum consisting of:

- (1) An A_∞ -algebra R .
- (2) An L_∞ -algebra \mathcal{L} .
- (3) An L_∞ -morphism $\alpha : \mathcal{L} \rightarrow R\text{Der}(R) = C^{\geq 1}(R, R)[1]$

is the same as an algebra differential d in the (completed) tensor product $S^\bullet(\mathcal{L}^*[-1]) \otimes T^\bullet(R^*[-1])$ which preserves $S_+^\bullet(\mathcal{L}^*[-1]) \otimes 1$ as well as $1 \otimes T_+^\bullet(R^*[-1])$.

The relations of the data (1) and (2) to the differential d has already been explained in the general proof of Proposition 7.5. Let us explain how to obtain the datum (3) explicitly.

We recall, first of all, the general meaning of the concept of an L_∞ -morphism $\beta : \mathcal{L} \rightarrow \mathcal{L}'$ between two L_∞ -algebras \mathcal{L} and \mathcal{L}' whose L_∞ -operations we denote by λ_n, λ'_n respectively. By definition, β is the same as a morphism of commutative dg-algebras

$$(7.7) \quad \beta^* : S^\bullet(\mathcal{L}'^*[1]) \longrightarrow S^\bullet(\mathcal{L}^*[1]),$$

the datum which spells out to a collection of morphisms of graded vector spaces

$$(7.8) \quad \beta_n : S^n(\mathcal{L}[-1]) \longrightarrow \mathcal{L}'[-1], \quad n \geq 0$$

satisfying a chain of compatibility conditions. More precisely, any collection of the β_n in (7.8) defines a unique morphisms of *commutative graded algebras* β^* as in (7.7) and the conditions on the β_n say that β^* thus defined, actually commutes with the differentials. Note, in particular, that β_1 can be seen, after shift, as a morphism of graded vector spaces $\beta_1[1] : \mathcal{L} \rightarrow \mathcal{L}'$ and the first in the chain of conditions on the β_n says that $\beta_1[1]$ commutes with the differentials λ_1 and λ'_1 in \mathcal{L} and \mathcal{L}' . This is the “underlying morphism” of the L_∞ -morphism β , and higher β_n can be seen as higher homotopies for its compatibility with the brackets.

We now specialize to our situation: $\mathcal{L}' = C^{\geq 1}(R, R)[1]$. Suppose we are given a differential d in $S^\bullet(\mathcal{L}^*[-1]) \otimes T^\bullet(R^*[-1])$ which preserves $S^\bullet(\mathcal{L}^*[-1])$. We describe the corresponding L_∞ -morphism α in terms of its components

$$\alpha_n : S^n(\mathcal{L}[-1]) \rightarrow C^{\geq 1}(R, R)[1] = \bigoplus_{m \geq 1} \text{Hom}_k(R^{\otimes m}, R)[1 - m].$$

More precisely, the matrix element

$$\alpha_{n,m} : S^n(\mathcal{L}[-1]) \rightarrow \text{Hom}_k(R^{\otimes m}, R)[1 - m]$$

of α_n with the respect to the above direct sum decomposition of the RHS is obtained as the dual of the matrix element of the restriction $d|_{R^*[-1]}$ which we denote

$$d_{n,m} : R^*[-1] \rightarrow S^n(\mathcal{L}^*[-1]) \otimes T^m(R^*[-1]), \quad T^m(R^*[-1]) = (R^*)^{\otimes m}[-m].$$

Example 7.9 (Particular case $\mathcal{P} = \text{Lie}$). A datum consisting of:

- (1) An L_∞ -algebra \mathcal{L}_∞ .
- (2) An L_∞ -algebra \mathcal{L} .
- (3) An L_∞ -morphism $\alpha : \mathcal{L} \rightarrow R\text{Der}_{\text{Lie}}(R) = C_{\text{Lie}}^{\geq 1}(\mathcal{L}, \mathcal{L})[1]$

is the same as an algebra differential d in the (completed) symmetric algebra

$$S^\bullet(\mathcal{L}^*[-1]) \otimes S^\bullet(\mathcal{L}_\infty^*[-1]) = S^\bullet((\mathcal{L} \oplus \mathcal{L}_\infty)^*[-1]),$$

preserving $S_+^\bullet(\mathcal{L}^*[-1]) \otimes 1$ as well as $1 \otimes S_+^\bullet(\mathcal{L}_\infty^*[-1])$. The correspondence between the matrix elements of d and the matrix elements of the components α_n is completely analogous to Example 7.6, and we leave it to the reader.

Note that the data of a differential d as above, is the same as an L_∞ -structure on $\mathcal{L} \oplus \mathcal{L}_\infty$ such that \mathcal{L} is an L_∞ -subalgebra and \mathcal{L}_∞ is an L_∞ -ideal. This is precisely the kind of situation we had in §6. So we obtain the following.

Corollary 7.10. *In the situation of §6, we have a natural L_∞ -morphism*

$$\alpha : \mathfrak{g} \longrightarrow R\mathrm{Der}_{\mathcal{L}ie}(\mathfrak{g}_\infty). \quad \square$$

8 Refinement in $d = 2$: relative setting.

Consider the relative setting of §6 of which we keep the notation: $\tilde{A} = A \cup \{\infty\}$ etc.

In this case the L_∞ -algebra \mathfrak{g}_∞ can be upgraded to an A_∞ -algebra R_∞ , similar to §5. This is because, for each purely infinite coarse subdivision $\mathcal{P}'' = \{(Q''_\nu, A''_\nu)\}$ of an infinite marked subpolygon (Q', A') , $\infty \in A'$, the infinite marked polygons Q''_ν , are naturally ordered using the chosen orientation of \mathbb{R}^2 and the projection $p : \mathbb{R}^2 - \{\infty\} \rightarrow \mathbb{R}$ as in Remark 6.3(a).

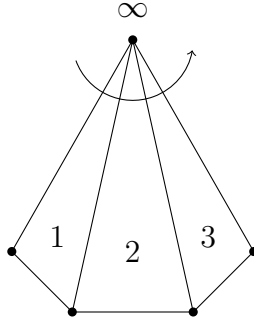


Figure 9: Ordering of infinite polygons in \mathbb{R}^2 .

So if we put

$$V_\infty = \bigoplus_{(Q', A') \text{ infinite}} V_{A'}$$

then the formulas for the chain differential in the $C_\bullet(\Sigma(A'))$ define an algebra differential d in $T^\bullet(V_\infty)$ with $d^2 = 0$. This means that $\dot{R}_\infty = V_\infty^*[-1]$ is an A_∞ -algebra.

Similarly to §5, $\dot{R}_\infty = \dot{\mathfrak{g}}_\infty$ as a graded vector space. Further, the only A_∞ -operations in R_∞ are the differential and the binary bracket, so R_∞ is an associative dg-algebra.

As before, we define the geometric subalgebra $R_\infty \subset \dot{R}_\infty$ spanned by geometric infinite marked polygons $(Q', A \cap Q')$. Thus $R_\infty = \mathfrak{g}_\infty$ as a vector space and the differential on R_∞ is trivial, so R_∞ is a graded associative algebra.

The product of two infinite marked subpolygons a, b is zero unless their union is again an infinite marked subpolygon (in particular, the union is

convex), with a being on the left of b and meeting b along an edge, see Fig. 10.

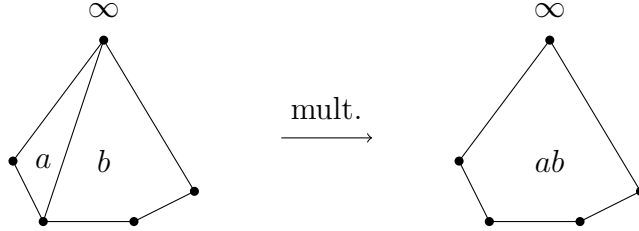


Figure 10: Multiplication on R_∞ .

Now, looking at all coarse subdivisions of all marked subpylgon of (\tilde{Q}, \tilde{A}) , both finite and infinite, we get an algebra differential in

$$S^\bullet(V) \otimes T^\bullet(V_\infty), \quad V = \bigoplus_{(Q', A') \text{ finite}} V_{A'}.$$

This differential preserves $S^\bullet(V)$ and gives there the differential defining \mathfrak{g}_∞ . So we obtain, by Proposition 7.5 and, more particularly, Example 7.6, L_∞ -morphisms

$$(8.1) \quad \dot{\phi} : \dot{\mathfrak{g}}_\infty \longrightarrow R\text{Der}_{\mathcal{A}ss}(\dot{R}_\infty), \quad \phi : \mathfrak{g}_\infty \longrightarrow R\text{Der}_{\mathcal{A}ss}(R_\infty),$$

of which the second one will be more important for us. This is the setting of Gaiotto-Moore-Witten [GMW] in the “case of trivial coefficients”. In the next sections we explain how can one generalize our setting to fully include that of [GMW].

9 Introducing coefficients: factorizing sheaves on secondary polytopes.

Since we assumed that $A \subset \mathbb{R}^d$ is in general position, all proper faces of Q are simplices. Each subset of d points $\{\omega_1, \dots, \omega_d\} \subset A$ gives a $(d-1)$ -simplex $\sigma = \text{Conv}\{\omega_1, \dots, \omega_d\}$.

By an *oriented simplex* we will mean a simplex together with a numbering of vertices considered up to an even permutation. We will use the notation $\sigma = \langle \omega_1, \dots, \omega_d \rangle$ to denote the oriented $(d-1)$ -simplex given by the numbering of the vertices $\omega_1, \dots, \omega_d$. We write $\bar{\sigma}$ for “ σ with the opposite orientation”, i.e., $\bar{\sigma} = \langle \omega_{s(1)}, \dots, \omega_{s(d)} \rangle$, where s is any odd permutation.

Definition 9.1. A *system of coefficients* for A is a datum $N = (N_\sigma)$ of cochain complexes, one for each oriented $(d-1)$ -simplex σ with vertices in A , so that $N_{\bar{\sigma}} = N_\sigma^*$.

We fix an orientation of \mathbb{R}^d . Then for each marked subpolytope $(Q', A') \subset (Q, A)$, each codimension 1 face of Q' is naturally an oriented $(d-1)$ -simplex. We denote

$$(9.2) \quad N_{A'} = \bigotimes_{\sigma \subset \partial Q'} N_\sigma.$$

Let $\mathcal{P} = \{(Q_\nu, A_\nu)\}$ be a regular polyhedral subdivision of (Q, A) . We define

$$(9.3) \quad N_{\mathcal{P}} = \bigotimes_{\nu} N_{A_\nu}.$$

If \mathcal{P}' is a refinement of \mathcal{P} , then we have morphisms of cochain complexes

$$\gamma_{\mathcal{P}'\mathcal{P}} : N_{\mathcal{P}'} \rightarrow N_{\mathcal{P}}$$

obtained by taking traces $N_\sigma \otimes N_{\bar{\sigma}} \rightarrow \mathbf{k}$ over those $(d-1)$ -faces of \mathcal{P}' which are *internal*, i.e., are not faces of \mathcal{P} and thus belong to exactly two polytopes from \mathcal{P}' .

Proposition 9.4. *The maps $\gamma_{\mathcal{P}'\mathcal{P}}$ are transitive for chains of refinements: if \mathcal{P}'' refines \mathcal{P}' and \mathcal{P}' refines \mathcal{P} , then*

$$\gamma_{\mathcal{P}''\mathcal{P}} = \gamma_{\mathcal{P}'\mathcal{P}} \circ \gamma_{\mathcal{P}''\mathcal{P}'}. \quad \square$$

We now interpret this proposition in terms of constructible complexes of sheaves on secondary polytopes. Let X be a CW-complex with a cellular stratification $(X_\alpha)_{\alpha \in I}$. In other words, (I, \leq) is a poset together with a strictly monotone function $d : I \rightarrow \mathbb{Z}_+$, each X_α is a locally closed subspace of X homeomorphic to the Euclidean space $\mathbb{R}^{d(\alpha)}$, and the condition $\alpha \leq \beta$ is equivalent to the X_α being contained in the closure of X_β . A sheaf \mathcal{F} of \mathbf{k} -vector spaces on X constant along each X_α is, as well known [Cur][GM] described by the following data:

- (1) Vector spaces $\mathcal{F}_\alpha = H^0(X_\alpha, \mathcal{F})$ (stalk of \mathcal{F}).
- (2) *Generalization maps* $\gamma_{\alpha\beta} : \mathcal{F}_\alpha \rightarrow \mathcal{F}_\beta$ defined for any $\alpha \leq \beta$ and satisfying the transitivity conditions, i.e., forming a representation of the poset I in \mathbf{k} -vector spaces.

Proposition 9.4 implies that the data $(N_{\mathcal{P}}, \gamma_{\mathcal{P}'\mathcal{P}})$ give rise to a complex of sheaves \mathcal{N}_A on the secondary polytope $\Sigma(A)$ which is constant on each open face $F_{\mathcal{P}}^\circ$ with stalk $N_{\mathcal{P}}$ and with the generalization maps being the $\gamma_{\mathcal{P}'\mathcal{P}}$. We then have the following counterpart of Proposition 1.9.

Proposition 9.5. *The complex \mathcal{N}_A satisfies the factorization property: the restriction of \mathcal{N}_A to any closed face $F_{\mathcal{P}} = \prod_{(Q'_i, A'_i) \in \mathcal{P}} \Sigma(A'_i)$ is identified with the exterior tensor product $\boxtimes_{\nu} \mathcal{N}_{A'_i}$. \square*

The fact that the generalization maps for a cellular sheaf go from the stalks at smaller cells to the stalks at bigger cells, is a part of the general phenomenon that sheaves have *cohomology* rather than *homology*. In particular, the cellular cochain complex of a sheaf \mathcal{F} given by $(\mathcal{F}_\alpha, \gamma_{\alpha\beta})$ has the form

$$C_{\text{cell}}^\bullet(X, \mathcal{F}) = \left\{ \bigoplus_{\dim X_\alpha=0} \text{or}(X_\alpha) \otimes \mathcal{F}_\alpha \longrightarrow \bigoplus_{\dim X_\alpha=1} \text{or}(X_\alpha) \otimes \mathcal{F}_\alpha \longrightarrow \dots \right\}$$

where $\text{or}(X_\alpha) = H_c^{d(\alpha)}(X_\alpha, \mathbf{k})$ is the 1-dimensional orientation space of the cell X_α and the differentials are induced by the $\gamma_{\alpha\beta}$.

So sticking to the cohomological picture, we form the graded vector space

$$V_N^{\text{coh}} = \bigoplus_{\substack{A' \subset A \\ |A'| \geq d+1}} V_{A'}^{\text{coh}}, \quad V_{A'}^{\text{coh}} = N_{A'} \otimes \text{or}(\Sigma(A'))[-\dim \Sigma(A')].$$

Then the cellular *cochain* complexes of the factorizing sheaves $\mathcal{N}_{A'}$ unite to give a *coalgebra differential* d , $d^2 = 0$, in the symmetric coalgebra $S^\bullet(V_N^{\text{coh}})$. This is an alternative way of saying that:

Proposition 9.6. (a) *The cellular cochain differentials of the $\mathcal{N}_{A'}$ give rise to an L_∞ -algebra structure on*

$$\dot{\mathfrak{g}}_N = \dot{\mathfrak{g}}_{A,N} = \bigoplus_{(Q',A') \subset (Q,A)} E_{A'}, \quad E_{A'} = N_{A'} \otimes \text{or}(\Sigma(A'))[-\dim \Sigma(A') - 1].$$

The L_∞ -algebra $\dot{\mathfrak{g}}_N$ is nilpotent.

(b) *The subspace*

$$\mathfrak{g}_N = \mathfrak{g}_{A,N} = \bigoplus_{Q' \subset (Q,A)} E_{A \cap Q'}$$

is an L_∞ -subalgebra in $\dot{\mathfrak{g}}_N$. □

As before \mathfrak{g}_N will be more important for us than $\dot{\mathfrak{g}}_N$.

Thus, the component \mathfrak{g}_N^1 (where Maurer-Cartan elements live) is

$$\mathfrak{g}_N^1 = \bigoplus_{Q' \subset (Q,A)} N_{A'}^{-\dim \Sigma(A')} \otimes \text{or}(\Sigma(A')), \quad A' = A \cap Q'.$$

In other words, an element of \mathfrak{g}_N^1 is a rule associating to each marked d -simplex $(Q', A') \subset (Q, A)$ an element of $N_{A'}^0$, to each circuit (Q', A') an element of $N_{A'}^{-1}$ and so on.

Relative case. Suppose A is extended to $\tilde{A} = A \cup \{\infty\}$, as in §6. Let \tilde{N} be a coefficient system for \tilde{A} . We then generalize Propositions 6.2 and Corollary 7.10 to:

Proposition 9.7. (a) *We have a semi-direct product decomposition*

$$\mathfrak{g}_{\tilde{N}} = \mathfrak{g}_N \ltimes \mathfrak{g}_{\tilde{N},\infty}$$

where $\mathfrak{g}_{N,\infty}$ is the direct sum of summands $E_{A'}$ corresponding to infinite marked subpolytopes (Q', A') . Therefore we have an L_∞ -morphism $\alpha : \mathfrak{g}_{\tilde{N}} \rightarrow R\text{Der}_{\mathcal{L}ie}(\mathfrak{g}_{\tilde{N},\infty})$.

(b) *If $d = 2$, then $\mathfrak{g}_{\tilde{N},\infty}$ lifts to an A_∞ -algebra $A_{\tilde{N},\infty}$ and we have an L_∞ -morphism $\psi : \mathfrak{g}_{\tilde{N}} \rightarrow R\text{Der}_{\mathcal{A}ss}(R_{\tilde{N},\infty})$. □*

10 Coefficients in bimodules: $d = 2$

The systems of coefficients considered in §9 consisted in data associated to codimension 1 simplices only. It would be natural to consider more general coefficient systems which would associate some data to simplices of all dimensions from 0 up to $(d - 1)$. In this section we explain how to do this in the case $d = 2$.

Let us denote elements of A as i, j, k and so on.

Definition 10.1. An *extended system of coefficients* for A is a following system of data:

- (1) For each $i \in A$, an associative dg-algebra S_i .
- (2) For each oriented edge (1-simplex) (i, j) , $i, j \in A$, a (differential graded) (S_i, S_j) -bimodule N_{ij} which we assume projective, of finite rank, over the graded algebra underlying $S_i \otimes_{\mathbf{k}} S_j^{\text{op}}$.
- (3) For each (i, j) , a pairing

$$\beta_{ij} : N_{ij} \otimes_{\mathbf{k}} N_{ji} \longrightarrow S_i \otimes_{\mathbf{k}} S_j,$$

which is a morphism of $(S_i \otimes_{\mathbf{k}} S_j, S_i \otimes_{\mathbf{k}} S_j)$ -bimodules. It is required that:

- (4) The pairing β_{ij} is non-degenerate, i.e., the morphism of (S_i, S_j) -bimodules

$$\beta_{ij}^t : N_{ij} \rightarrow \text{Hom}_{S_j \otimes_{\mathbf{k}} S_i^{\text{op}}}(N_{ji}, S_j \otimes_{\mathbf{k}} S_i^{\text{op}})$$

induced by β_{ij} , is an isomorphism.

- (5) The diagram

$$\begin{array}{ccc} N_{ij} \otimes_{\mathbf{k}} N_{ji} & \xrightarrow{\beta_{ij}} & S_i \otimes_{\mathbf{k}} S_j \\ \text{perm.} \downarrow & & \downarrow \text{perm.} \\ N_{ji} \otimes_{\mathbf{k}} N_{ij} & \xrightarrow{\beta_{ji}} & S_j \otimes_{\mathbf{k}} S_i \end{array}$$

is commutative.

Thus the setting of Definition 9.1, specialized to $d = 2$, corresponds to the case when all $S_i = \mathbf{k}$.

Remark 10.2. The data of an extended system of coefficients can be interpreted as follows. The datum (1) means that we have a (pre-)triangulated dg-category \mathcal{C}_i . More precisely, let us assume that S_i is smooth and proper [TV], in particular, that $\dim_{\mathbf{k}} S_i < \infty$. Then we take $\mathcal{C}_i = D \text{Mod}_{S_i}^{<\infty}$ to be the derived category formed by finite-dimensional right dg-modules over S_i . Note that \mathcal{C}_i has a Serre functor [BK], i.e., a covariant exact functor $\mathfrak{S}_i : \mathcal{C}_i \rightarrow \mathcal{C}_i$ equipped with natural isomorphisms

$$\text{Hom}_{\mathcal{C}_i}(M, M')^* \simeq \text{Hom}_{\mathcal{C}_i}(M', \mathfrak{S}_i(M)).$$

Here and elsewhere $*$ means dualization over \mathbf{k} . Explicitly, \mathfrak{S}_i is given by the Nakayama formula

$$(10.3) \quad \mathfrak{S}_i(M) = (R \text{Hom}_{S_i}(M, S_i))^*.$$

The datum (2) means that we have an exact functor $F_{ij} : \mathcal{C}_i \rightarrow \mathcal{C}_j$ (given by the tensor product with N_{ij}). Finally, the datum (3) means that F_{ij} is left adjoint to $\mathfrak{S}_i F_{ji}$ (or, equivalently, that F_{ji} is left adjoint to $\mathfrak{S}_j F_{ij}$). This follows from (10.3).

Note that for a pair of exact functors $F : \mathcal{C} \rightarrow \mathcal{D}$, $G : \mathcal{D} \rightarrow \mathcal{C}$ between triangulated categories \mathcal{C}, \mathcal{D} with Serre functors $\mathfrak{S}_{\mathcal{C}}, \mathfrak{S}_{\mathcal{D}}$ the condition “ F is left adjoint to $\mathfrak{S}_{\mathcal{C}} G$ ” is symmetric in F and G (unlike the usual condition of adjointness). Indeed, such “twisted adjointness” is expressed via natural isomorphisms

$$\text{Hom}_{\mathcal{D}}(F(X), Y)^* \simeq \text{Hom}_{\mathcal{C}}(G(Y), X), \quad X \in \mathcal{C}, Y \in \mathcal{D}.$$

We fix an orientation of \mathbb{R}^2 . Suppose given an extended system of coefficients (S_i, N_{ij}) . Let $Q' \subset (Q, A)$ be a subpolygon. Let us label the vertices of Q' cyclically, in the counterclockwise order as i_0, \dots, i_m . We can then form the *cyclic tensor product*

$$N_{Q'} = \left(N_{i_0 i_1} \otimes_{S_{i_1}} N_{i_1 i_2} \otimes_{S_{i_2}} \cdots \otimes_{S_{i_{m-1}}} N_{i_{m-1} i_m} \right) \otimes_{S_{i_0}^{\text{op}} \otimes S_{i_m}} N_{i_m i_0},$$

which can be pictorially represented in Figure 11, which makes evident its cyclic symmetry (independence on the choice of the starting point i_0). Note that $N_{Q'}$ is just a cochain complex: all module structures have been used in the formation of the tensor products.

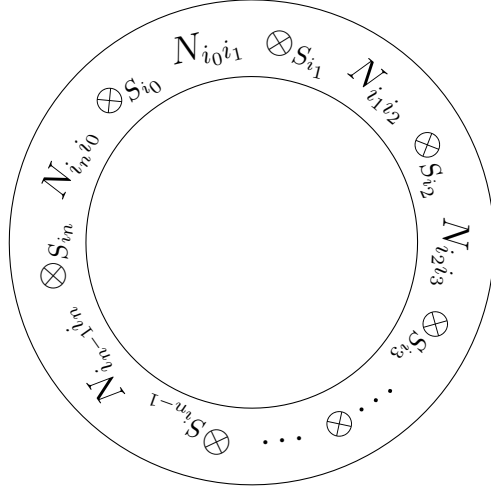


Figure 11: The cyclic tensor product.

Remark 10.4. More generally, we can define a cyclic tensor product N_σ for any closed oriented edge path $\sigma = (i_0, i_1, \dots, i_n, i_0)$, not necessarily coming from the boundary of a convex polygon. As before, N_σ depends only on σ as a closed path and not on the choice of a starting point.

Let

$$\sigma_1 = (\dots, h, i, j, k, \dots), \quad \sigma_2 = (\dots, p, j, i, q, \dots)$$

be two closed oriented edge paths having one edge $[i, j]$ in common, with opposite orientations. Then we can concatenate σ_1 and σ_2 along this edge, erasing it and obtaining a new closed oriented edge path

$$\sigma_1 *_{[i,j]} \sigma_2 = (\dots, p, j, k, \dots, h, i, q, \dots),$$

see Fig. 12.

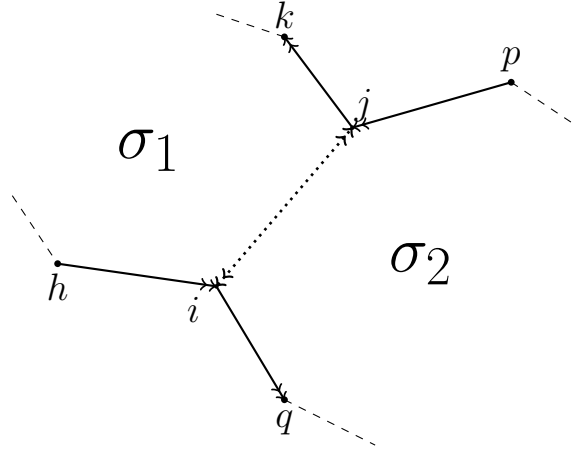


Figure 12: Concatenation.

In this situation, the pairing β_{ij} gives rise to the *concatenation map*

$$(10.5) \quad \gamma_{\sigma_1, \sigma_2}^{[i, j]} : N_{\sigma_1} \otimes_{\mathbf{k}} N_{\sigma_2} \longrightarrow N_{\sigma_1 *_{[i, j]} \sigma_2}.$$

To define it, consider two decomposable tensors

$$\begin{aligned} n_{\sigma_1} &= \cdots \otimes n_{hi} \otimes n_{ij} \otimes n_{jk} \otimes \cdots \in N_{\sigma_1}, \\ n_{\sigma_2} &= \cdots \otimes n_{pj} \otimes n_{ji} \otimes n_{iq} \otimes \cdots \in N_{\sigma_2}, \end{aligned}$$

where $n_{ij} \in N_{ij}$ etc. Suppose

$$\beta_{ij}(n_{ij} \otimes n_{ji}) = \sum_{\nu} s'_{\nu} \otimes s''_{\nu}, \quad s'_{\nu} \in S_i, \quad s''_{\nu} \in S_j.$$

Then we define

$$\gamma_{\sigma_1, \sigma_2}^{[i, j]}(n_{\sigma_1} \otimes n_{\sigma_2}) = \sum_{\nu} (\cdots \otimes n_{pj} s''_{\nu} \otimes n_{jk} \otimes \cdots \otimes n_{hi} s'_{\nu} \otimes n_{iq} \otimes \cdots).$$

It is immediate to check that this gives a well defined map of cyclic tensor products.

If now \mathcal{P} is a polygonal subdivision of $Q' = \text{Conv}(A')$ into $\{Q''_{\nu}\}$, we define

$$N_{\mathcal{P}} = \bigotimes_{\nu} N_{Q''_{\nu}}$$

(tensor product over \mathbf{k}). Note that the concatenation maps (10.5) corresponding to various intermediate edges of \mathcal{P} , commute with each other, so the result of applying them all in any order, is independent on the order. We call this result the *composition map* and denote it

$$(10.6) \quad \gamma_{\mathcal{P}} : N_{\mathcal{P}} \longrightarrow N_{A'}.$$

Further, if \mathcal{P}' is a subdivision refining \mathcal{P} , the composition maps (10.6) corresponding to all polygons of \mathcal{P} , give rise to a map

$$(10.7) \quad \gamma_{\mathcal{P}'\mathcal{P}} : N_{\mathcal{P}'} \longrightarrow N_{\mathcal{P}}$$

which we call the *generalization map* to emphasize that we want to use it to construct cellular sheaves on secondary polytopes. More precisely, we have the following statement whose proof is straightforward.

Proposition 10.8. *The data $(N_{\mathcal{P}}, \gamma_{\mathcal{P}'\mathcal{P}})$ define a factorizing system of complexes of sheaves $\mathcal{N}_{A'}$ on the $\Sigma(A')$, with the stalk of $\mathcal{N}_{A'}$ on the face $F_{\mathcal{P}} \subset \Sigma(A')$ being $N_{\mathcal{P}}$. Denote*

$$E_{A'} = N_{A'} \otimes \text{or}(\Sigma(A'))[-\dim \Sigma(A') - 1].$$

Then the differentials in the cellular cochain complexes of the $\mathcal{N}_{A'}$ unite to make the graded vector space $\mathring{\mathfrak{g}} = \bigoplus_{(Q', A') \subset (Q, A)} E_{A'}$ into a nilpotent L_{∞} -algebra. The subspace $\mathfrak{g} = \bigoplus_{Q' \subset (Q, A)} E_{A \cap Q'}$ is an L_{∞} -subalgebra in $\mathring{\mathfrak{g}}$. \square

Relative situation (as in §6). $\tilde{A} = A \cup \{\infty\}$, $\tilde{Q} = \text{Conv}(\tilde{A})$. Let (S_i, N_{ij}) be an extended coefficient system for A (not \tilde{A}) For any infinite marked polytope (Q', A') with vertices ∞, i_0, \dots, i_m in counterclockwise order, we define the *linear tensor product*

$$L_{A'} = N_{i_0 i_1} \otimes_{S_{i_1}} N_{i_1 i_2} \otimes_{S_{i_2}} \cdots \otimes_{S_{i_{m-1}}} N_{i_{m-1} i_m}.$$

This is an (S_{i_0}, S_{i_m}) -bimodule. If \mathcal{P} is a subdivision of a (possibly infinite) subpolygon (Q', A') into (Q''_{ν}, A''_{ν}) , then we define

$$N_{\mathcal{P}} = \bigotimes_{A''_{\nu} \not\ni \infty} N_{A''_{\nu}} \otimes \bigotimes_{A''_{\nu} \ni \infty} L_{A''_{\nu}}.$$

and these complexes, again, give a factorizable system of complexes of sheaves on the $\Sigma(A')$. Restricting to the geometric summands as before, we obtain:

Proposition 10.9. Denoting for $A' \ni \infty$,

$$F_{A'} = L_{A'} \otimes \text{or}(\Sigma(A'))[-\dim \Sigma(A') - 1],$$

we get the following data:

- (1) An L_∞ -algebra $\mathfrak{g} = \bigoplus_{Q' \neq \infty} E_{A \cap Q'}$.
- (2) An A_∞ -algebra $R_\infty = \bigoplus_{Q' \ni \infty} F_{A \cap Q'}$.
- (3) An L_∞ -morphism $\psi : \mathfrak{g} \rightarrow R\text{Der}(R_\infty)$.

□

Remark 10.10. Study of Fukaya-Seidel categories (see §14 below for more background) leads naturally to a structure somewhat more general and symmetric than an extended system of coefficients. In that structure, for each point $i \in A$ we have not just a category, but a *local system of triangulated categories* (together with pre-triangulated dg-enhancements) \mathcal{C}_i over a small circle S_i^1 centered at i . It is required that:

- (a) Each fiber category $\mathcal{C}_{i,\theta}$, $\theta \in S_i^1$ has Serre functor \mathfrak{S}_i .
- (b) The monodromy around the circle S_i^1 gives rise to the functor isomorphic to $\mathfrak{S}_i \circ [d]$, where $[d]$ denotes the functor of shift of the grading by $d \in \mathbf{Z}$ (for the Fukaya-Seidel category associated with the pair (X, W) the number d is the dimension of the complex manifold X).

Further, for each pair of different points $i, j \in A$ we have functors $F_{ij} : \mathcal{C}_{j,\theta_j} \rightarrow \mathcal{C}_{i,\theta_i}$ and $F_{ji} : \mathcal{C}_{i,\theta_i} \rightarrow \mathcal{C}_{j,\theta_j}$ between the fiber categories over the points $\theta_i \in S_i^1$ and $\theta_j \in S_j^1$ which are intersection points of the straight segment ij with circles S_i^1 and S_j^1 . These functors are required to be “twisted adjoint” to each other in the sense of the Remark 10.2.

A choice of a “generic” point at infinity (equivalently, of a generic half-plane) breaks the local rotational symmetry, since each circle S_i^1 contains the intersection point with the ray from i to infinity. This allows us to replace each S_i^1 by a segment. After this, the more general structure described above reduces effectively to an extended system of coefficients.

11 Analysis of the algebra R_∞ and the morphism ψ for $d = 2$.

Since we assumed that $\tilde{A} = A \cup \{\infty\}$ is in general position, A has a total order by the slope of the line $(i\infty)$ from left to right. We get a decomposition

$$R_\infty = \bigoplus_{\substack{i,j \in A \\ i < j}} R_{ij},$$

where R_{ij} is the direct sum of the summands $F_{A \cap Q'}$ corresponding to infinite polygons Q' whose edges through ∞ are $(i\infty)$ and $(j\infty)$. Note that R_{ij} is an (S_i, S_j) -bimodule.

Proposition 11.1. (a) *The higher compositions $m_n, n \geq 3$, in R_∞ , are trivial. The only non-trivial part of the binary compositions m_2 are the maps*

$$\mu_{ijk} : R_{ij} \otimes R_{jk} \rightarrow R_{ik}, \quad i < j < k.$$

Thus R_∞ is a strictly upper-triangular dg-algebra (without unit).

(b) *The multiplication μ_{ij} is S_j -bilinear with respect to the right S_j -module structure on R_{ij} and the left S_j -module structure on R_{jk} . Therefore, putting $R_{ii} = S_i$, we obtain a non-strictly triangular associative dg-algebra with unit*

$$R = \bigoplus_{i \leq j} R_{ij}.$$

Remark 11.2. Note that part (b) means that Mod_R has a semi-orthogonal decomposition with quotients

$$\text{Mod}_{S_{i_1}}, \text{Mod}_{S_{i_2}}, \dots, \text{Mod}_{S_{i_r}},$$

where $i_1 < i_2 < \dots < i_r$ are all the elements of A in our order.

Proof of Proposition 11.1: (a) The only coarse subdivisions of an infinite polygon into all infinite subpolygons contain either 2 parts (and so account for m_2), or 1 part (and so account for $m_1 = d$, as in Examples 1.8 (c2) and (c1). The subdivisions into 1 part are excluded in the geometric subalgebra $R_\infty \subset \dot{R}_\infty$. This shows part (a). Part (b) is clear. \square

We now describe the morphism

$$\psi : \mathfrak{g} \longrightarrow R\mathrm{Der}(R_\infty) = C^{\geq 1}(R_\infty, R_\infty)[1]$$

from Proposition 10.5 explicitly. Because of the direct sum decompositions

$$\mathfrak{g} = \bigoplus_{A' \subset A \text{ geom.}} E_{A'}, \quad R_\infty = \bigoplus_{B \ni \infty \text{ geom.}} F_B, \quad |A'|, |B| \geq 3,$$

ψ is given by its matrix elements with respect to the induced direct sum decompositions of the source and target. These matrix elements are morphisms of dg-vector spaces

$$(11.3) \quad \psi_{A'}^{(B_1, \dots, B_m | C)} : E_{A'} \longrightarrow \mathrm{Hom}_{\mathbf{k}}(F_{B_1} \otimes_{\mathbf{k}} \cdots \otimes_{\mathbf{k}} F_{B_m}, F_C),$$

given for all finite geometric $A' \subset A$ and all infinite geometric $B_1, \dots, B_m, C \subset A$ of cardinality ≥ 3 . By the analysis of Example 7.6, $\psi_{A'}^{(B_1, \dots, B_m | C)} \neq 0$ only if $Q' = \mathrm{Conv}(A')$ together with the $P_\nu = \mathrm{Conv}(B_\nu)$, forms a coarse regular subdivision of $P = \mathrm{Conv}(C)$. In other words, nonzero matrix elements of ψ correspond to coarse regular subdivisions of infinite subpolygons in (\tilde{Q}, \tilde{A}) which contain exactly one finite polygon (the rest being infinite). Let us call such subdivisions *1-finite*. We will now describe all 1-finite subdivisions explicitly, starting with the following class of examples.

Example 11.4. Let Q' be a finite subpolygon of (Q, A) , and let

$$A' = A \cap Q', \quad \tilde{Q}' = \mathrm{Conv}(A' \cup \{\infty\}), \quad \tilde{A}' = \tilde{A} \cap \tilde{Q}'.$$

Since we assumed that no 3 points of \tilde{A} lie on a line, the boundary $\partial Q'$ splits into two distinct parts meeting at two points:

- The *positive boundary*, i.e., the union of sides facing ∞ , drawn thick in Figure 13 and denoted $\partial_+ Q'$.
- The *negative boundary*, denoted $\partial_- Q'$ formed by the sides which lie on the opposite side of ∞ .

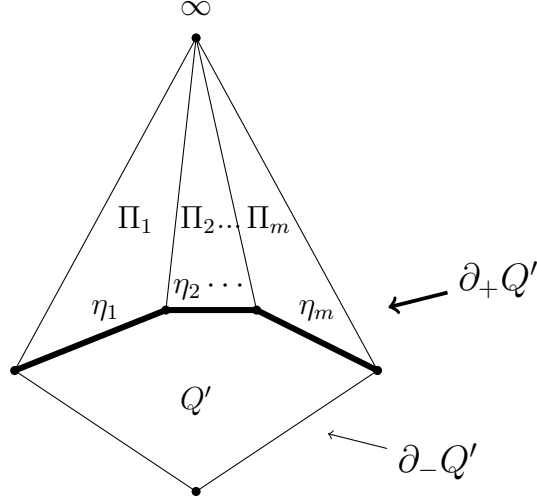


Figure 13: The positive and negative boundary.

We denote the sides of $\partial_+ Q'$ by η_1, \dots, η_m in the natural order from left to right (counter-clockwise rotation around ∞), and let Π_1, \dots, Π_m be the triangles with one vertex at ∞ and the opposite sides being η_1, \dots, η_m . Let also $D_\nu = \tilde{A} \cap \Pi_\nu$. Then (Π_ν, D_ν) is an infinite marked subpolygon of (Q', A') , and we have a polygonal subdivision (see Fig. 13)

$$(11.5) \quad (\tilde{Q}', \tilde{A}') = (Q', A') \cup (\Pi_1, D_1) \cup \dots \cup (\Pi_m, D_m).$$

It is easy to see that (11.5) is a 1-finite subdivision. Indeed, the unique (up to scalar factors and adding global affine-linear functions) piecewise-affine function $f : \tilde{Q}' \rightarrow \mathbb{R}$ is determined by putting $f(\infty) = 1$ and $f|_{A'} = 0$.

However, Example 11.4 does not give the most general form of a 1-finite subdivision. To obtain the general form, considering the following situation, extending the above one.

Example 11.6. Let $P \subset (\tilde{Q}, \tilde{A})$ be an infinite subpolygon and $Q' \subset P$ be a finite subpolygon such that $\partial_- Q'$ is contained in ∂P and is the union of edges from ∂P . The two remaining finite parts of ∂P (not in $\partial_- Q'$) will be called *handles*, see Fig. 14. We denote the left handle by λ and edges constituting it by $\lambda_1, \dots, \lambda_p$. Similarly, we denote the right handle by ρ and the edges constituting it by ρ_1, \dots, ρ_q , in our order from left to right. The handles may be empty, i.e., it may happen that $p = 0$ or $q = 0$ or both.

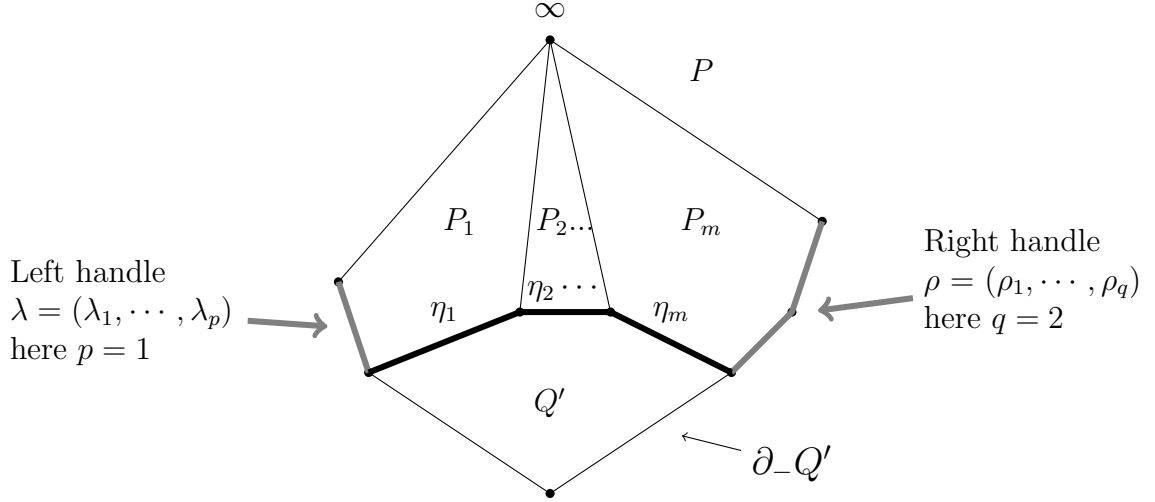


Figure 14: General 1-finite subdivision.

Note that by taking the convex hull \tilde{Q}' of Q' and ∞ , we get the situation described in Example 11.4, from which we retain the notation. We then have a decomposition of P into Q' and polygons P_1, \dots, P_m , where:

- $P_\nu = \Pi_\nu$ for $\nu = 2, \dots, m - 1$.
- The finite part of ∂P_1 is composed out of λ and η_1 .
- The finite part of ∂P_m is composed out of η_m and ρ .

Proposition 11.7. *The subdivision described in Example 11.6, is 1-finite, and each 1-finite subdivision is obtained in this way.*

Proof: By construction, the subdivision contains exactly one finite subpolygon. Let us show that it is regular. For this, note that, as in Example 11.4, the conditions $f(\infty) = 1$ and $f_{Q'} = 0$ determine a unique continuous function $f : P \rightarrow \mathbb{R}$ which is affine on each polygon and breaks exactly along the subdivision. Further, this function is unique with these properties up to scalar factors and adding global affine functions. This follows from analyzing the restriction to $\tilde{Q}' = \text{Conv}(Q \cup \{\infty\})$. Therefore our subdivision is coarse and therefore is 1-finite.

Conversely, note that the construction of Example 11.6 by its very nature produces all subdivisions of any infinite polygon P which contain exactly one finite subpolygon. \square

In the situation of Example 11.6, we put

$$A' = A \cap Q', \quad B_\nu = \tilde{A} \cap P_\nu, \quad C = \tilde{A} \cap P.$$

In order to describe the corresponding matrix element $\psi_{A'}^{(B_1, \dots, B_m | C)}$, we write it in the transposed way, as a map

$$(11.8) \quad \psi_{A'; B_1, \dots, B_m}^C : E_{A'} \otimes_{\mathbf{k}} F_{B_1} \otimes_{\mathbf{k}} \cdots \otimes_{\mathbf{k}} F_{B_m} \longrightarrow F_C.$$

Let us denote

$$N_\lambda = N_{\lambda_1} \otimes_S \cdots \otimes_S N_{\lambda_p}, \quad N_\rho = N_{\rho_1} \otimes_S \cdots \otimes_S N_{\rho_q}$$

the linear tensor products corresponding to the handles λ and ρ considered as oriented edge paths. Here and in the sequel we write \otimes_S for the tensor product over the algebra S_i corresponding to the vertex common to the two edges in an edge path. Then

$$F_{B_1} = N_\lambda \otimes_S E_{\eta_1}, \quad F_{B_\nu} = E_{\eta_\nu}, \quad 2 \leq \nu \leq m-1, \quad F_{B_m} = E_{\eta_m} \otimes_S N_\rho, \\ F_C = N_\lambda \otimes_S E_{A'} \otimes_S N_\rho.$$

Let

$$(11.9) \quad \gamma_{Q'} : E_{A'} \otimes_{\mathbf{k}} F_{D_1} \otimes_{\mathbf{k}} \cdots \otimes_{\mathbf{k}} F_{D_m} \longrightarrow F_{\tilde{A}'}$$

be the composition map obtained, similarly to (10.6), by substituting the pairings $\beta_{i,j}$ over all the intermediate edges $[i, j]$ in the coarse subdivision (11.5). Define

$$(11.10) \quad \psi_{A'; B_1, \dots, B_m}^C = \text{Id}_{N_\lambda} \otimes_S \gamma_{Q'} \otimes_S \text{Id}_{N_\rho}.$$

Then the source and target of $\psi_{A'; B_1, \dots, B_m}^C$ are as specified in (11.8).

Proposition 11.11. *The maps $\psi_{A'; B_1, \dots, B_m}^C$ corresponding to all pairs (P, Q') consisting of an infinite subpolygon $P \subset (\tilde{Q}, \tilde{A})$ and a finite subpolygon Q' with $\partial_- Q'$ contained in the finite part of ∂P , are precisely all the matrix elements of the L_∞ -morphism $\psi : \mathfrak{g} \rightarrow C^{\geq 1}(R_\infty, R_\infty)[1]$ with respect to the direct sum decompositions of the source and target described above.*

Proof: This is a consequence of the analysis of Example 7.6 applied to our case. \square

12 The universality theorem ($d = 2$).

We work in the setting of §10 and 11. So we have the triangular dg-algebra

$$R = \bigoplus_{i \leq j} R_{ij}, \quad R_{ii} = S_i.$$

We are interested in deformations of R which, first, preserve the triangular structure and, second, do not change the quotients of the semi-orthogonal decomposition. The second condition means that the algebras S_i are not deformed. Such deformations are governed by the *ordered Hochschild complex* $\vec{C}^\bullet(R, R)$, with

$$\begin{aligned} \vec{C}^n(R, R) &= \\ &= \bigoplus_{i_0 < i_1 < \dots < i_n} \text{Hom}_{S_{i_0} \otimes S_{i_n}^{\text{op}}} \left(R_{i_0 i_1} \otimes_{S_{i_1}} R_{i_1 i_2} \otimes_{S_{i_2}} \dots \otimes_{S_{i_{n-1}}} R_{i_{n-1} i_n}, R_{i_0 i_n} \right) \end{aligned}$$

(strict inequalities under the direct sum sign).

Remark 12.1. Note that $\vec{C}^\bullet(R, R)$ is a subcomplex in the ordinary Hochschild complex $C^\bullet(R, R)$, specified by the multi-linearity conditions. Moreover, this subcomplex is closed with respect to the Lie bracket and is thus a dg-Lie sub-algebra.

Theorem 12.2. *The L_∞ -morphism ψ factors through an L_∞ -morphism*

$$\Psi : \mathfrak{g} \longrightarrow \vec{C}^{\geq 1}(R, R)[1].$$

The morphism Ψ is a quasi-isomorphism.

Remarks 12.3. (a) Note that the differential in \mathfrak{g} is trivial, while that in $\vec{C}^{\geq 1}(R, R)[1]$ is not.

(b) Theorem 12.2 suggests an alternative way to think about the L_∞ -algebra \mathfrak{g} of convex polygons. Namely, we can start with the dg-algebra R and define the L_∞ -structure on \mathfrak{g} by transferring, along the quasi-isomorphism Ψ , the DGLA structure from $\vec{C}^\bullet(R, R)$ to a L_∞ -structure on \mathfrak{g} . Explicit formulas for the transfer in terms of sums over planar trees can be written similarly to those from [KoSo3], Section 4. Note that the transferred structure a priori depends on a choice of ∞ and hence is different from the one described

in Section 9. Because of our theorem, these two L_∞ structures are equivalent. In particular, the transferred structure does not, up to equivalence, depend on a choice of ∞ .

On the other hand, the transferred structure has the advantage that it can be defined in terms of the *oriented matroid* generated by the set \tilde{A} , i.e., by the knowledge of which pairs consisting of a point $\omega \in \tilde{A}$ and a subset $A' \subset \tilde{A}$, satisfy $\omega \in \text{Conv}(A')$, see [OM]. This is a much weaker and more combinatorial datum, in particular, it cannot be used to recover the concepts of a regular decomposition or a convex piecewise affine function.

Proof of the theorem: For simplicity, let us write $\vec{\mathcal{C}}^\bullet$ for $\vec{\mathcal{C}}^{\geq 1}(R, R)[1]$ in the sequel.

Let us prove the first statement of the theorem. By Proposition 11.11, the components of ψ are the maps $\psi_{A'}^{(B_1, \dots, B_m | C)}$ from (11.3) and (11.10) and we keep the corresponding notation. These are non-zero only if F_{B_1}, \dots, F_{B_m} are summands of $R_{i_0 i_1}, R_{i_1 i_2}, \dots, R_{i_{m-1} i_m}$ and F_C is a summand of $R_{i_0 i_m}$. They are also bilinear with respect to the intermediate algebras. This proves that ψ factors through a morphism Ψ with values in $\vec{\mathcal{C}}^\bullet$.

Let us now prove that Ψ is a quasi-isomorphism. It will be convenient to organize the proof into several steps.

A. Interpretation of $\vec{\mathcal{C}}^\bullet$ via closed paths. Recall that $R_{ij} = \bigoplus_{Q'} F_{A \cap Q'}$ where Q' runs over infinite subpolygons with edges $[\infty, i]$ and $[j, \infty]$.

Definition 12.4. A sequence P_0, P_1, \dots, P_n of marked infinite polygons is called *admissible*, if its left and right infinite edges match as in Fig. 15, that is, there are $i_0 < \dots < i_n$ such that P_0 has edges $[\infty, i_0]$ and $[i_n, \infty]$ which P_ν has edges $[\infty, i_{\nu-1}]$ and $[i_\nu, \infty]$. Note that it is not required that P_0 actually contains any of the P_ν .

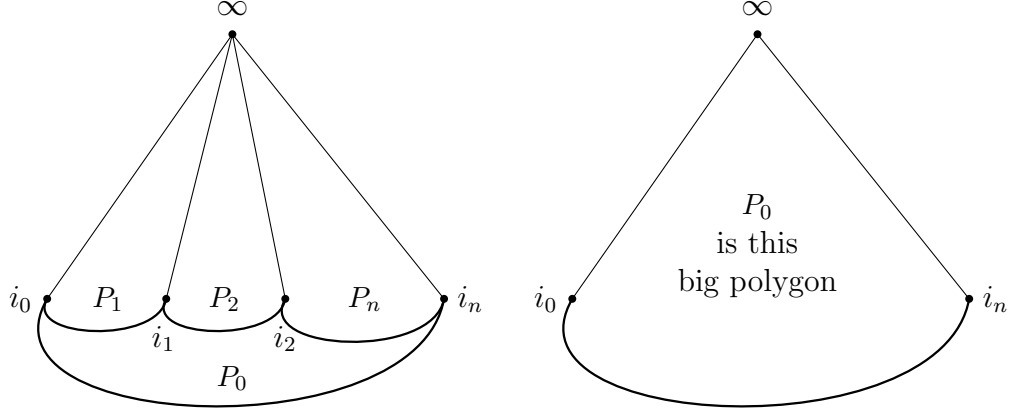


Figure 15: An admissible sequence.

We denote $B_\nu = \tilde{A} \cap P_\nu$. For an admissible sequence P_0, P_1, \dots, P_n we define

$$F_{P_1, \dots, P_n}^{P_0} = \text{Hom}_{S_{i_0} \otimes S_{i_n}^{\text{op}}} \left(F_{B_1} \otimes_{S_{i_1}} F_{B_2} \otimes_{S_{i_2}} \cdots \otimes_{S_{i_{n-1}}} F_{B_n}, F_{B_0} \right).$$

Then

$$\vec{\mathcal{C}}^n = \bigoplus_{P_0, P_1, \dots, P_n \text{ admiss.}} F_{P_1, \dots, P_n}^{P_0}.$$

We note further that $F_{P_1, \dots, P_n}^{P_0}$ is the same as the cyclic tensor product N_σ (in the sense of Remark 10.4) over the closed edge path σ obtained by running over the negative boundaries $\partial_- P_1, \partial_- P_2, \dots, \partial_- P_n$ and then over $\partial_- P_0$ in the opposite direction, see Fig. 16.

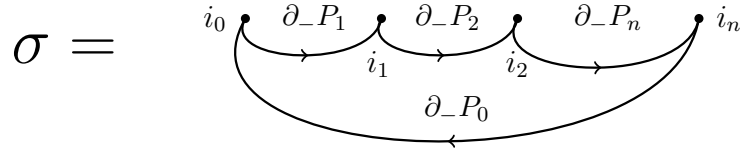


Figure 16: $F_{P_1, \dots, P_n}^{P_0}$ as a cyclic tensor product N_γ .

B. Filtration of $\vec{\mathcal{C}}^\bullet$ by handle length. Note that closed edge paths σ labeling summands in $\vec{\mathcal{C}}^\bullet$, can retrace parts of themselves on the left or

on the right. Similarly to §11, we will call such retraced parts *handles* and denote them by λ and ρ , see Fig. 17.

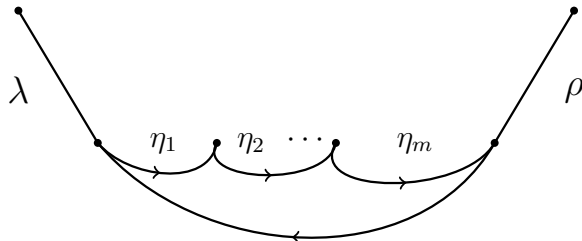


Figure 17: A closed path with handles.

Lemma 12.5. *Put*

$$G^l \vec{C}^\bullet = \bigoplus_n \bigoplus F_{P_1, \dots, P_n}^{P_0},$$

the second sum running over admissible sequences P_0, P_1, \dots, P_n with the sum of edge lengths of the two “handles” on the left and right being $\geq l$. Then G is a decreasing filtration of \vec{C}^\bullet by subcomplexes.

Proof: The differential in \vec{C}^\bullet is the sum $d + \delta$, where δ is the Hochschild differential, induced by the multiplication in R , and d is induced by the differential in R itself. We recall the formula for

$$\delta : \text{Hom}(R^{\otimes n}, R) \longrightarrow \text{Hom}(R^{\otimes(n+1)}, R)$$

(in our case, Hom and \otimes are taken over intermediate algebras, as explained above):

$$\begin{aligned} (\delta f)(a_1, \dots, a_{n+1}) &= a_1 f(a_2, \dots, a_{n+1}) + \\ &+ \sum_{\nu=1}^n (-1)^\nu f(a_1, \dots, a_\nu a_{\nu+1}, \dots, a_{n+1}) + (-1)^n f(a_1, \dots, a_n) a_{n+1}. \end{aligned}$$

Recall also the formula for the multiplication m_2 in R :

$$m_2|_{F_{B_1} \otimes F_{B_2}} = \begin{cases} \text{The canonical map } F_{B_1} \otimes F_{B_2} \rightarrow F_{B_1 \cup B_2}, & \text{if} \\ Q_1 \cup Q_2 \text{ is convex and } Q_1 \text{ is on the left of } Q_2; \\ 0, & \text{otherwise.} \end{cases}$$

Here Q_1 and Q_2 are two infinite subpolygons, and $B_\nu = \tilde{A} \cap Q_\nu$.

Consider a summand $F_{P_1, \dots, P_n}^{P_0} \in \vec{\mathcal{C}}^n(R, R)$. The first and last terms in the formula for δ , applied to this summand, consist in adding an infinite polygon P to the picture on the left and on the right so that $\partial_- P$ together with $\partial_- P_0$ form a convex polygon (if the union is not convex, the contribution is 0). See Fig. 18.

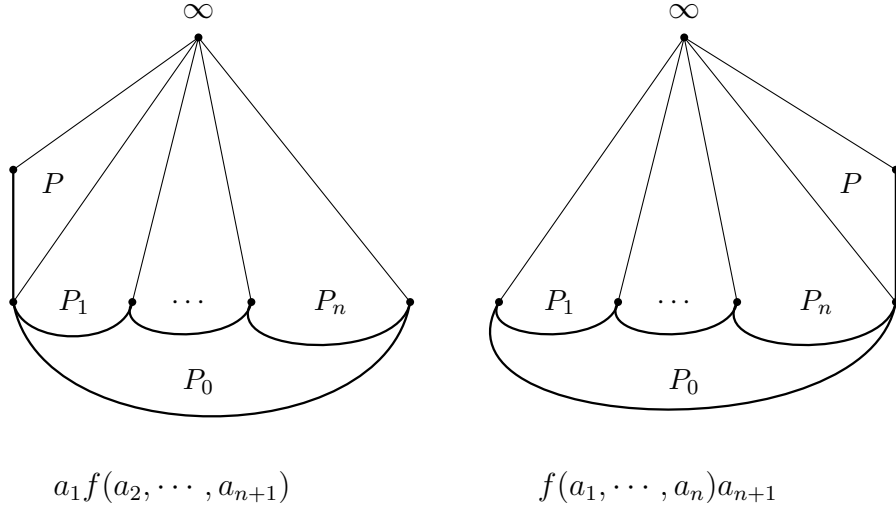
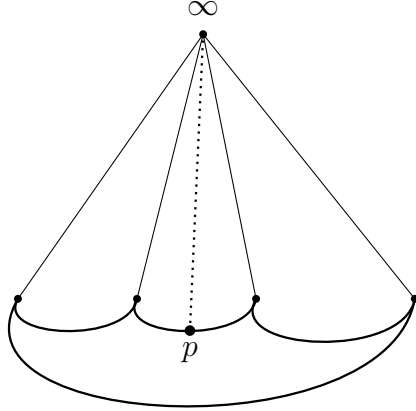


Figure 18: The first and last terms in δ .

The new summands in which these terms lie, correspond to closed edge paths with the length of the left or right handle increased. The remaining summands $(-1)^\nu f(a_1, \dots, a_\nu a_{\nu+1}, \dots, a_{n+1})$ of δ send $F_{P_1, \dots, P_n}^{P_0}$ into summands obtained by *all possible splittings* of each of the polygons P_ν , $\nu = 1, \dots, n$, into two subpolygons along some edge of the form $[\infty, p]$ where p is some intermediate vertex. If P_ν is a triangle and cannot be split, then the corresponding contribution is 0. See Fig. 19. These summands have the same lengths of handles. Therefore δ preserves the filtration G , and so does $d + \delta$. \square



$$f(a_1, \dots, a_\nu a_{\nu+1}, \dots, a_{n+1})$$

Figure 19: Intermediate terms in δ .

C. $\bar{\Psi} : \mathfrak{g} \rightarrow \text{gr}^0 \vec{\mathcal{C}}^\bullet$ is an embedding with exact quotient. Consider the morphism of complexes

$$\bar{\Psi} : \mathfrak{g} \longrightarrow \text{gr}_G^0 \vec{\mathcal{C}}^\bullet = \vec{\mathcal{C}}^\bullet / G^1 \vec{\mathcal{C}}^\bullet,$$

induced by Ψ . We can think of $\text{gr}_G^0 \vec{\mathcal{C}}^\bullet$ as the direct sum of cyclic tensor products N_σ corresponding to closed paths σ without handles (and coming from admissible sequences).

On the other hand, \mathfrak{g} is the direct sum of cyclic tensor products taken over the boundaries of (convex) finite subpolygons Q' . These boundaries are particular cases of the paths we obtain for $\text{gr}_G^0 \vec{\mathcal{C}}^\bullet$. Further, it follows from Proposition 11.11 that the only matrix element of $\bar{\Psi}$ on $E_{Q'}$ not landing in $G^1 \vec{\mathcal{C}}^\bullet$, is the transpose of the map $\gamma_{Q'}$ from (11.9). This transpose is in fact an isomorphism (identity)

$$\gamma_{Q'}^t : E_{A'} \longrightarrow N_{\partial Q'} = E_{A'}.$$

This establishes the following.

Lemma 12.6. $\bar{\Psi}$ is an embedding of complexes, whose image is the direct sum of summands $F_{P_1, \dots, P_n}^{P_0}$ such that the negative boundaries $\partial_- P_\nu$, $\nu = 1, \dots, n$, are segments which, together with $\partial_- P_0$, form a convex polygon. \square

So it is enough to prove that $\text{Coker}(\overline{\Psi})$ is exact. As a vector space, $\text{Coker}(\overline{\Psi})$, is simply the direct sum of all the summands $F_{P_1, \dots, P_n}^{P_0}$ without handles, other than those forming a convex polygon as above.

Now, in gr_F the first and last terms of the formula for δ disappear (they increase the sum of the lengths of the handles). The remaining intermediate terms correspond to splitting each P_ν , $\nu = 1, \dots, n$ along possible intermediate vertices p of P_ν as in Fig. 19:

$$P_\nu \rightsquigarrow P'_\nu \cup P''_\nu.$$

The summand corresponding to the split configuration is identical with the original one:

$$F_{P_1, \dots, P_{\nu-1}, P'_\nu, P''_\nu, P_{\nu+1}, \dots, P_n}^{P_0} = F_{P_1, \dots, P_n}^{P_0}.$$

Thus $\text{gr}_F \overrightarrow{C}^\bullet$ is split into a direct sum of complexes of the form

$$F_{P_1, \dots, P_n}^{P_0} \otimes_{\mathbf{k}} C^\bullet(\Delta^{r-1}),$$

where:

- (a) P_0, P_1, \dots, P_n is an admissible sequence such that none of the intermediate adjacent pairs $P_\nu, P_{\nu+1}$, $i + 1 \leq n$, can be combined together to form a convex polygon (from which they could be obtained by splitting).
- (b) r is the number of intermediate vertices of P_1, \dots, P_n .
- (c) Δ^{r-1} is the $(r - 1)$ -dimensional combinatorial simplex whose vertices correspond to the intermediate vertices (the ways of splitting) in (b), and $C^\bullet(\Delta^{r-1})$ is the augmented simplicial cochain complex of Δ^{r-1} with coefficients in \mathbf{k} . Note that $C^\bullet(\Delta^{r-1})$ is exact if $r \geq 0$.

So the remaining summands $F_{P_1, \dots, P_n}^{P_0}$ which will not be tensored with an exact complex, correspond to admissible sequences P_0, P_1, \dots, P_n which:

- (1) Cannot be split anywhere, i.e., each P_ν , $\nu = 1, \dots, n$, is a triangle, and so $\partial_- P_\nu$ is a segment.
- (2) Cannot appear in the splitting of something else. This means that the segments $\partial_- P_1, \dots, \partial_- P_n$ form an *upwardly convex* broken line from i_0 to i_n , which, together with $\partial_- P_0$ (a *downwardly convex* broken line)

form a convex finite polygon Q' representing a summand in $\Psi(\mathfrak{g})$, see Fig. 20, where $\partial_- P_0$ depicted curved to emphasize that it can consist of many segments.

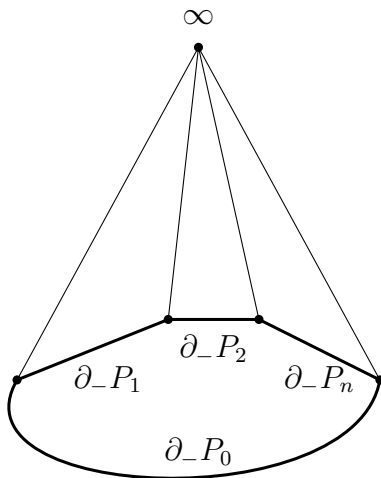


Figure 20: A convex polygon appearing.

These are precisely the summands forming $\overline{\Psi}(\mathfrak{g})$. This proves that $\overline{\Psi}$ is a quasi-isomorphism.

D. All $\text{gr}^{\geq 1} \overrightarrow{C}^\bullet$ are exact. To finish the proof of Theorem 12.2, it will now be sufficient to show that each $\text{gr}_G^l \overrightarrow{C}^\bullet$, $l \geq 1$, is an exact complex.

This is done similarly to Step C. Since the first and last summands in δ disappear in gr_G , the differential induced by δ on $\text{gr}_G^l \overrightarrow{C}^\bullet$ consists in splittings of the polygons P_ν into two subpolygons, as in Fig. 19. So $\text{gr}_G^l \overrightarrow{C}^\bullet$ splits into a direct sum of sub complexes corresponding to different total shapes of the pictures, and the differential in each such summand consists of splittings (the δ part) plus the differential d induced from d_R .

Now, if $l \geq 1$, the picture representing each summand has at least one handle, λ or ρ , nontrivial. If λ is nontrivial, then the edge path σ representing the picture allows for nontrivial splittings of the union of λ and η_1 , see Fig. 17. As in part C, this will exhibit a nontrivial (i.e., contractible) $C^\bullet(\Delta^{r-1})$ as a tensor factor of our summand. Similarly if ρ is nontrivial.

Theorem 12.2 is proved.

13 Comparison with Gaiotto-Moore-Witten.

In this section we provide a dictionary between the approach and terminology of [GMW] and the ones we use in the main body of the paper.

Vacua = elements of A . The construction of [GMW] starts with a finite set \mathbb{V} of “vacua” whose elements are denoted by i, j, k , etc., and a “weight map” $z : \mathbb{V} \rightarrow \mathbb{C}$ (which we assume injective). Our set A is the image of this map. So in this section we denote elements of A by $z_i, i \in \mathbb{V}$.

Webs=polygonal decompositions. We recall:

Definition 13.1. ([GMW]) A *plane web* is a graph \mathfrak{w} in the plane $\mathbb{R}^2 = \mathbb{C}$ together with a labeling of faces (“countries”) by vacua so that:

- (1) Different faces have different labels.
- (2) If an edge is oriented so that i is on the left and j on the right, then this edge is parallel to the vector $z_i - z_j$ (i.e., has the same oriented direction but possibly different nonzero length).

Remark 13.2. More precisely, in [GMW] the labels are allowed to be repeated, with (1) replaced by a weaker condition: labels differ across edges. Such repeating patterns are accounted for by taking the symmetric algebra as in §2, and we do not consider them here.

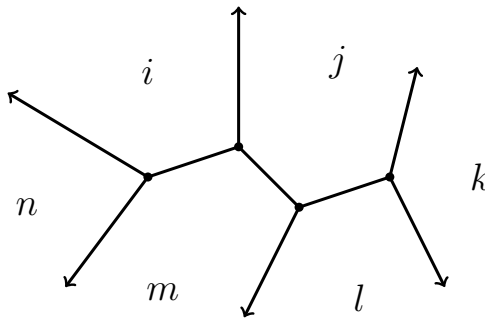


Figure 21: A plane web. Edges with arrows go to infinity.

We call the set of vacua that appear as labels of the faces of a web \mathfrak{w} , the *support* of \mathfrak{w} . We further recall that two webs \mathfrak{w} and \mathfrak{w}' are said to be of the

same *deformation type*, if they are topologically equivalent in the following restricted sense: \mathfrak{w}' can be obtained from \mathfrak{w} by translation and stretching (but not rotating) of the edges. Following [GMW], we denote by $D(\mathfrak{w})$ the set formed by all webs of the same deformation type as \mathfrak{w} , considered modulo overall translations. This set is called the *moduli space* of webs in the fixed deformation type.

Proposition 13.3. *Let $A' \subset A$ and $Q' = \text{Conv}(A')$. Then:*

- (a) *Deformation types of webs with support A' are in bijection with regular geometric polygonal decompositions \mathcal{P}' of (Q', A') , i.e., with geometric faces $F_{\mathcal{P}'}$ of the secondary polytope $\Sigma(A')$.*
- (b) *If \mathfrak{w} is a web whose deformation type corresponds to a decomposition \mathcal{P}' , then the moduli space $D(\mathfrak{w})$ is identified with the reduced normal cone $\overline{\mathcal{C}}_{\mathcal{P}'}$ to the face $F_{\mathcal{P}'} \subset \Sigma(A')$, see §1.*

Proof: Given a polygonal decomposition $\mathcal{P}' = \{(Q''_\nu, A''_\nu)\}$ of (Q', A') , we form the dual graph by putting one vertex in each polygon Q''_ν and joining them as in Fig. 22:

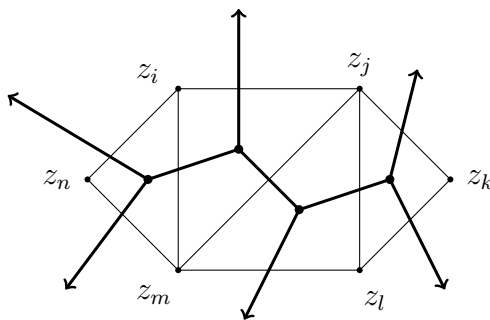


Figure 22: Webs are dual to decompositions.

This gives a plane graph Γ with faces labelled by elements of A' but not necessarily satisfying the condition (2) of Definition 13.1. We now show that possible ways of deforming Γ to a web, modulo translations, are in bijection with $\overline{\mathcal{C}}_{\mathcal{P}'}$.

We first recall that $\overline{\mathcal{C}}_{\mathcal{P}'}$ consists of convex \mathcal{P}' -piecewise affine functions $Q' \rightarrow \mathbb{R}$, considered modulo adding global affine functions on Q' , i.e., $\overline{\mathcal{C}}_{\mathcal{P}'} =$

$C_{\mathcal{P}'}/\text{Aff}(\mathbb{R}^2)$, where $C_{\mathcal{P}'}$ consists of actual convex \mathcal{P}' -piecewise affine functions.

Next, we note that rotation by 90° transforms the condition (2) of Definition 13.1 into:

- (2') An edge having i on the left and j on the right, is *perpendicular* to the vector $z_i - z_j$ (i.e., forms, together with this vector, a positively oriented orthogonal frame).

So we will work with (2') instead of (2). Let us identify $(\mathbb{R}^2)^*$ with \mathbb{R}^2 by means of the standard scalar product.

Suppose given a piecewise-linear $f \in C_{\mathcal{P}'}$ (an actual function, not a class modulo adding global affine functions). For each polygon $Q''_\nu \in \mathcal{P}'$ let $p_\nu \in (\mathbb{R}^2)^*$ be the slope of the affine function $f|_{Q''_\nu}$. By the above, we can regard p_ν as a point of \mathbb{R}^2 . If two polygons $Q''_\mu, Q''_\nu \in \mathcal{P}'$ have a common edge, the slopes of f must match on this edge, This simply means that $p_\mu - p_\nu$ is perpendicular to the edge.

So we obtain a plane graph \mathfrak{w} topologically equivalent to Γ by joining p_μ and p_ν by a straight line internal whenever Q''_μ and Q''_ν have a common edge. By the above, \mathfrak{w} satisfies (2'), i.e., it is a (90° rotation of a) web. In this way the condition (2') expresses the existence of a global piecewise affine function f with the slope p_ν on each Q''_ν . The function f is defined by the knowledge of the p_ν up to an additive constant. Further, the *convexity* of f is expressed by saying that the edges of \mathfrak{w} (with respect to our choice of orientations) have *positive length* (a concave break would result in the length of an edge counted as negative). Translating \mathfrak{w} as a whole amounts to adding a global linear function to f . We leave the remaining details to the reader. \square

We now briefly indicate the meaning, in our terms, of some further terminology of [GMW].

Cyclic fans of vacua = subpolygons $Q' = \text{Conv}(A')$, $A' \subset A$. Indeed, according to *loc. cit.*, a cyclic fan of vacua is a cyclically ordered set $I = \{i_1, \dots, i_n\}$ such that the rays $(z_{i_k} - z_{i_{k+1}})\mathbb{R}_+$ are oriented clockwise. This simply corresponds to z_{i_1}, \dots, z_{i_n} being the vertices of a convex n -gon, in this cyclic order.

Further, if a web \mathfrak{w} corresponds to a regular polygonal subdivision \mathcal{P}' of (Q', A') then vertices of \mathfrak{w} are labeled, by the above, by polygons Q''_ν of \mathcal{P}' , and the local fan of \mathfrak{w} at a vertex v , denoted $I_v(\mathfrak{w})$, corresponds the polygons Q''_ν labeling v . The fan at infinity, denoted $I_\infty(\mathfrak{w})$, corresponds to Q' itself.

Rigid, taut and sliding webs. They correspond to faces of a secondary polytope $\Sigma(A')$, $A' \subset A$ of codimension 0, 1, and 2. This is a consequence of Proposition 13.3.

Convolution of webs. It corresponds to further subdivision of one polygon of a polygonal decomposition:

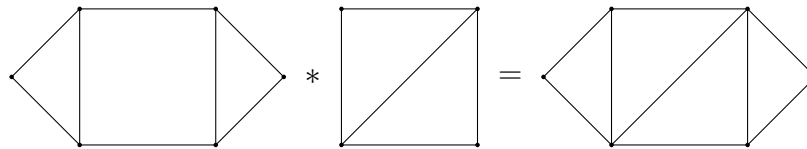


Figure 23: Convolution of webs in dual language.

Half-plane webs. They correspond, in the dual language, to polygonal subdivisions of the extended polytope $\tilde{Q} = \text{Conv}(\tilde{A})$ where $\tilde{A} = A \cup \{\infty\}$, see Figs. 8 and 24. The choice of the direction towards ∞ corresponds to the choice of a half-plane H in [GMW]. In Fig. 24, H is positioned horizontally.

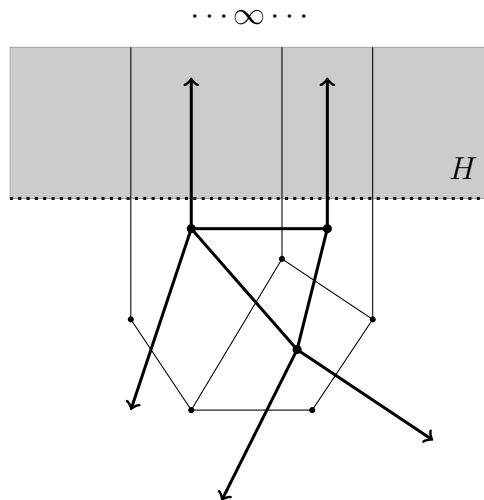


Figure 24: A half-plane web in the dual language.

14 Maurer-Cartan elements for the Fukaya-Seidel category

In this section we sketch, using our approach, an application of the previous considerations to the Fukaya-Seidel (FS) categories. We assume familiarity with the conventional approach to the subject [Se1], so our reminder in that respect will be minimal.

A. Setup. Let X be a complex Kähler manifold X of complex dimension k . The Kähler metric, denoted $\omega^{1,1}$, can be understood as a family of complex valued Hermitian scalar products in the \mathbb{C} -vector spaces $T_x X$. Globally, it gives a smooth isomorphism of \mathbb{C} -vector bundles

$$(14.1) \quad \omega^{1,1} : \overline{TX} \longrightarrow T^*X,$$

where \overline{TX} is the conjugate bundle to the \mathbb{C} -vector bundle TX . Separating the real and imaginary parts, we write $\omega^{1,1} = g + \omega\sqrt{-1}$. Thus g is a Riemannian metric and ω is a symplectic form on X . We also assume that X is equipped with a nowhere vanishing holomorphic volume form $\Omega^{k,0}$ (to be used later to obtain \mathbb{Z} -gradings of various Floer complexes).

Let $W : X \rightarrow \mathbb{C}$ be a holomorphic function with only non-degenerate (Morse) critical points. These points correspond to vacua of the Landau-Ginzburg theory associated to W . We denote the set of critical points by \mathbb{V} and assume it finite: $\mathbb{V} = \{x_1, \dots, x_r\}$. We denote by $w_i = W(x_i)$ the critical value at x_i and put $A = \{w_1, \dots, w_r\} \subset \mathbb{C}$. We assume that A is in (affine) general position: no three points lie on a line. By identifying \mathbb{C} with \mathbb{R}^2 , we can apply the formalism of previous sections to this choice of A . In particular, we have the polygon $Q = \text{Conv}(A)$ and can speak about its triangulations, its secondary polytope etc.

The real part $\text{Re}(W) : X \rightarrow \mathbb{R}$ is a Morse function on X (as a real $2k$ -dimensional manifold), with the same critical points x_1, \dots, x_r . Each x_i is Morse with the same signature (k, k) . As usual in Morse theory, by using the Riemannian metric g , we associate to x_i the *unstable manifold* (also known in this context as the *Lefschetz thimble*) $T_i = T_i(W) \simeq \mathbb{R}^k \subset X$. That is, T_i is the union of downward gradient trajectories $\phi(t)$ originating from x_i , i.e., such that $\lim_{t \rightarrow -\infty} \phi(t) = x_i$. We recall the following well known

Proposition 14.2. (a) *The gradient flow of $\text{Re}(W)$ with respect to g is equal to the Hamiltonian flow of $\text{Im}(W)$ with respect to ω .*

(b) Each T_i is Lagrangian with respect to ω . □

Since every Hamiltonian flow preserves the Hamiltonian, part (a) implies that the images $W(T_i)$ lie on horizontal half-lines $w_i + \mathbb{R}_{\leq 0} \subset \mathbb{C}$. See Fig. 25, where we also indicated the hyperplane H describing the “direction towards infinity”, cf. Fig. 24. Its boundary is now positioned vertically,

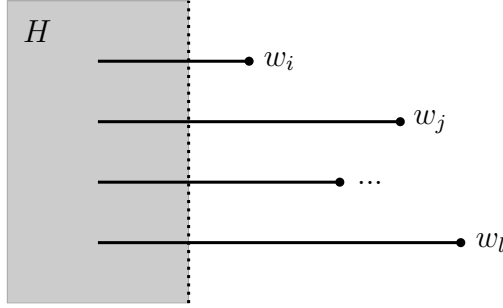


Figure 25: Thimbles project to half-lines.

Definition 14.3. The set A is called *horizontally generic*, if no two points of A have the same imaginary part ¹. For an angle θ we say that A is θ -generic, if $e^{\sqrt{-1}\theta}A$ is horizontally generic.

We assume that A is horizontally generic (which can be always assured by replacing W with some $e^{\sqrt{-1}\theta}W$). Then the half-lines $w_i + \mathbb{R}_{\leq 0}$ are disjoint, and so the thimbles T_i are disjoint closed Lagrangian submanifolds of X , which we assume numbered in order of increasing imaginary parts: $\text{Im}(w_1) < \dots < \text{Im}(w_r)$. We recall that \mathbf{k} is our fixed base field of characteristic 0. The *Fukaya-Seidel category* $\mathcal{FS} = \mathcal{FS}(X, W)$ is a \mathbf{k} -linear A_∞ -category in which T_1, \dots, T_r are objects, forming an exceptional collection:

$$\text{Hom}_{\mathcal{FS}}^\bullet(T_i, T_i) = \mathbf{k}, \quad \text{Hom}_{\mathcal{FS}}^\bullet(T_i, T_j) = 0, \quad i > j.$$

We denote

$$\mathcal{RS} = \mathcal{RS}(X, W) = \bigoplus_{i \leq j} \text{Hom}_{\mathcal{FS}}^\bullet(T_i, T_j)$$

the total A_∞ -algebra associated to \mathcal{FS} . The goal is to obtain \mathcal{RS} by deforming an appropriate algebra R as in §11 with respect to an appropriate Maurer-Cartan element.

¹In other words, the extended set $\tilde{A} = A \cup \{-\infty\}$ is in general position, see §6 and Fig.8.

Remark 14.4. There is no doubt that the formalism of Fukaya-Seidel categories generalizes to a situation much more general than the one described above: we must be able to allow W to have non-Morse (and even non-isolated) critical points. In this case, each critical value w_i (their number will still be finite in any algebraic situation) would give rise to a “local Fukaya-Seidel category” \mathcal{C}_i . These local categories would then embed into a global category \mathcal{FS} and form there a semi-orthogonal family. We expect that this more general setting to be still analyzable by the methods of this paper. In particular, the generality of extended systems of coefficients in §10 is (intentionally) geared to this future context, the category \mathcal{C}_i corresponding to the dg-algebra S_i , see Remark 10.2.

B. The system of coefficients. The algebra R comes from an extended coefficient system (S_i, N_{ij}^\bullet) as in §10. We put $S_i = \mathbf{k}$ (so the setting of §9 is in fact sufficient for this section). This corresponds to the fact that each x_i is a Morse singular point for W . The graded vector spaces N_{ij}^\bullet are obtained by analyzing those angles θ for which W is *not* θ -generic. These are precisely

$$\theta_{ij} = -\arg(w_i - w_j) \in S^1 = \mathbb{R}/2\pi\mathbb{Z}, \quad i \neq j.$$

We associate to them the unit vectors

$$\zeta_{ij} = e^{\sqrt{-1}\theta_{ij}} = \left(\frac{w_i - w_j}{|w_i - w_j|} \right)^{-1}.$$

Note that for any $\theta \in S^1$ the function $\operatorname{Re}(e^{\sqrt{-1}\theta}W) : X \rightarrow \mathbb{R}$ is again a Morse function with critical points x_1, \dots, x_n of signature (k, k) and so gives rise to the “ θ -rotated” thimbles $T_i^\theta = T_i(e^{\sqrt{-1}\theta}W)$. The image of T_i^θ under W lies now on the half-line originating from w_i and having angle $-\theta$ with the real axis. Following [GMW] we introduce the following

Definition 14.5. (a) By a ζ_{ij} -soliton we will mean a gradient trajectory $\phi = \phi(t)$ of $\operatorname{Re}(\zeta_{ij}W)$ originating (at $t = -\infty$) from x_i and terminating (at $t = +\infty$) at x_j . Thus the image of a ζ_{ij} -soliton under W is the straight interval $[w_i, w_j]$.

(b) By a *gradient polygon* we mean a sequence $\phi = (\phi_{i_0 i_1}, \phi_{i_1 i_2}, \dots, \phi_{i_m i_0})$, where $\phi_{i_\nu i_{\nu+1}}$ is a $\zeta_{i_\nu i_{\nu+1}}$ -soliton. A gradient polygon is called *convex*, or a *cyclic fan of solitons*, if the intervals $[w_{i_\nu}, w_{i_{\nu+1}}]$ form the boundary of a convex polygon, in the counter-clockwise order.

Thus the union of all ζ_{ij} -solitons is the intersection of the thimbles $T_{ij}^{\theta_{ij}}$ and $T_j^{\theta_{ji}}$. Since existence of gradient trajectories between critical points of the same index is a codimension 1 phenomenon in the space of Morse functions, see, e.g., [KS], the number of ζ_{ij} -solitons is “typically” finite. We assume this to be the case and denote this number by $n_{ij} = n_{ji}$. Further, let N_{ij} be the \mathbf{k} -vector space having, as basis vectors, symbols e_ϕ corresponding to the ζ_{ij} -solitons ϕ . So $\dim(N_{ij}) = n_{ij}$.

We now explain how to introduce a \mathbb{Z} -grading on each N_{ij} . As usual with Floer complexes (of which the N_{ij} are particular cases, see below), the grading is not fully canonical. More canonical are the following two types of constraints, which admit a solution but not uniquely.

Duality: $N_{ji} = N_{ij}^*$ as a graded space. Cf. the requirement (4) of Definition 10.1 .

Maslov index grading of cyclic products: The grading in each tensor product of the form $N_{i_0 i_1} \otimes \cdots \otimes N_{i_{m-1} i_m} \otimes N_{i_m i_0}$, $m \geq 1$ (cf. Fig. 11, where all $S_i = \mathbf{k}$), is given topologically as follows. By definition, this tensor product has a basis $\{e_\phi\}$ labelled by gradient polygons ϕ , and we define the degree of each e_ϕ . This is done by an instance of the Maslov index construction, which we review here in the generality that we need, referring to [Se1], §11 for the general case.

For a topological space Y and $x, y \in Y$ we denote by $\Pi_{x,y} Y$ the set of homotopy classes of paths in Y beginning at x and ending at y , i.e., the set of morphisms from x to y in the fundamental groupoid of X .

For a symplectic \mathbb{R} -vector bundle E over some base B we denote by $\text{LG}(E)$ the bundle of the Lagrangian Grassmannians of the fibers of E over B .

Consider first the case when $B = \text{pt}$, so E is a symplectic vector space. Let $L_1, L_2 \in \text{LG}(E)$ be two Lagrangian subspaces meeting transversely, i.e., $E = L_1 \oplus L_2$. In this case there is a canonical homotopy class $\kappa_{L_1, L_2} \in \Pi_{L_1, L_2} \text{LG}(E)$, defined uniquely by three properties:

- (1) Naturality with respect to symplectic isomorphisms $E \rightarrow E'$.
- (2) Additivity in symplectic direct sums. That is, if $E = E' \oplus E''$, and $L_\nu = L'_\nu \oplus L''_\nu$, then κ_{L_1, L_2} is the image of $(\kappa_{L'_1, L'_2}, \kappa_{L''_1, L''_2})$ under the morphism of fundamental groupoids induced by $\text{LG}(E') \times \text{LG}(E'') \hookrightarrow \text{LG}(E)$.

- (3) Normalization in dimension 2. If $\dim_{\mathbb{R}}(E) = 2$, then $\text{LG}(E) \simeq \mathbb{R}^1 = S^1$ is a circle, equipped with orientation. In this case κ_{L_1, L_2} is represented by the circle arc going from L_1 to L_2 in the positive direction.

Let us call κ_{L_1, L_2} the *symplectic bridge* between L_1 and L_2 .

Returning now to our situation, each gradient polygon ϕ gives a piecewise continuous closed path ϖ_{ϕ} in $\text{LG}(TX)$ obtained by associating to any internal point p of the segment $\phi_{i_{\nu}, i_{\nu+1}}$ the tangent space at p to the thimble $T_{i_{\nu}}(\zeta_{i_{\nu}, i_{\nu+1}}W)$. At each point i_{ν} , the path ϖ_{ϕ} has a simple discontinuity (jump), with the two limit values before and after the jump forming a pair of transversal Lagrangian subspaces in $T_{x_{\nu}}X$. Joining each such pair by the symplectic bridge, we obtain a closed continuous path ϖ_{ϕ}^{\wedge} in $\text{LG}(TX)$, defined uniquely up to homotopy.

Taking the exterior power and applying the holomorphic volume form $\Omega^{k,0}$ gives a continuous map (“phase”)

$$\text{LG}(TX) \xrightarrow{\Lambda_{\mathbb{R}}^k} \text{LG}(\Lambda_{\mathbb{C}}^k TX) \xrightarrow{\Omega^{k,0}} \text{LG}(\mathbb{C}) = \mathbb{R}P^1 = S^1.$$

Denote this composite map Θ . Note that $\text{LG}(\mathbb{C}) = S^1$ has a canonical counterclockwise orientation.

Definition 14.6. The Maslov index $d(\phi)$ of a gradient polygon ϕ is the winding number of the continuous closed path $\Theta(\varpi_{\phi}^{\wedge})$ in $S^1 = \text{LG}(\mathbb{C})$. We assign to the basis vector e_{ϕ} the degree $d(\phi)$.

C. Transversality assumptions. Note that the intersection of the Lagrangian manifolds $T_{ij}^{\theta_{ij}}$ and $T_j^{\theta_{ji}}$ is never transversal: if non-empty, it has dimension at least one. There are two ways of realizing n_{ij} as the number of intersection points of two Lagrangian manifolds which can be (and, “typically”, are) transversal.

- (1) Take an intermediate point p on the interval $[w_i, w_j]$ and consider the (smooth) Kähler manifold $W^{-1}(p)$ of dimension $k-1$. The intersections

$$V_i(p) = T_i^{\theta_{ij}} \cap W^{-1}(p), \quad V_j(p) = T_j^{\theta_{ji}} \cap W^{-1}(p)$$

are Lagrangian spheres (“vanishing cycles”) in $W^{-1}(p)$ whose intersection points are in bijection with trajectories constituting $T_{ij}^{\theta_{ij}}$ and $T_j^{\theta_{ji}}$.

See Fig. 26. We assume that $V_i(p)$ and $V_j(p)$ intersect transversally in $W^{-1}(p)$.

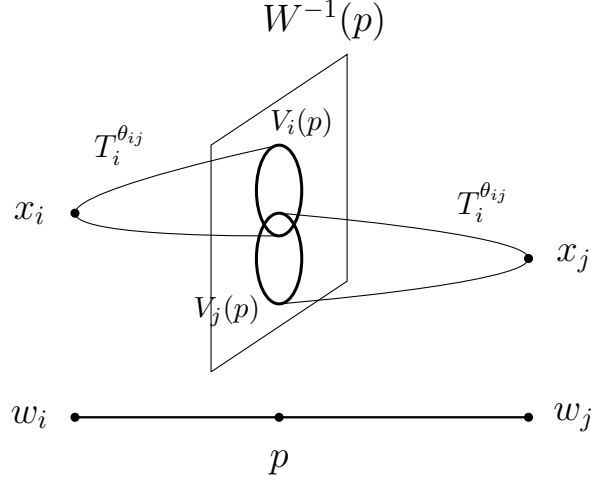


Figure 26: Vanishing cycles over a midpoint.

- (2) For small $\epsilon > 0$ put $\theta' = \theta_{ij} - \epsilon$ and $\theta'' = \theta_{ji} + \epsilon$. Then the thimbles $T_i^{\theta'}$ and $T_j^{\theta''}$ project by W onto two half lines which intersect in a point q somewhere near the interval $[w_i, w_j]$, see Fig. 27. The intersection points of $T_i^{\theta'}$ and $T_j^{\theta''}$, all lying over q , are again in bijection with trajectories constituting $T_{ij}^{\theta_{ij}} \cap T_j^{\theta_{ji}}$. We assume (this assumption is equivalent to that in (1)) that $T_i^{\theta'}$ and $T_j^{\theta''}$ intersect transversally in X .

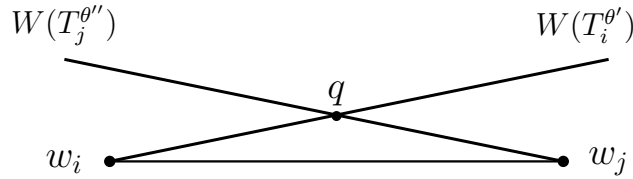


Figure 27: Perturbed thimbles in projection to \mathbb{C} .

D. The instanton equation. Suppose that the Kähler manifold X is exact, i.e., there exists a 1-form λ primitive for ω , i.e., $d\lambda = \omega$. It is convenient

to write λ symbolically as $\lambda = \mathbf{p} d\mathbf{q}$ which refers to a particular choice of a primitive in local Darboux coordinates $\mathbf{p} = (p_1, \dots, p_k)$, $\mathbf{q} = (q_1, \dots, q_k)$.

The ζ_{ij} -solitons, being Hamiltonian flow curves for $H = \text{Im}(\zeta_{ij}W)$ are, according to elementary Hamiltonian mechanics, critical points of the “phase space action” which is the functional

$$h_{\zeta_{ij}}(\phi) = \int_{\phi} \mathbf{p} d\mathbf{q} + H dt := \int_{\mathbf{R}} \phi^* \lambda + \text{Im}(\zeta_{ij}W) dt.$$

It is defined on the space \mathcal{X}_{ij} of smooth maps $\phi = \phi(t) : \mathbf{R} \rightarrow X$ with the boundary conditions

$$\lim_{t \rightarrow -\infty} \phi(t) = x_i, \quad \lim_{t \rightarrow +\infty} \phi(t) = x_j.$$

This interpretation is used in [GMW] for constructing both L_{∞} and A_{∞} structures in terms of supersymmetric quantum mechanics. That is, our transversality assumptions imply that $h_{\zeta_{ij}}$ is itself a Morse function on \mathcal{X}_{ij} , i.e., the ζ_{ij} -solitons are non-degenerate critical points. The grading on N_{ij} is, up to shift, given by the relative Morse (Floer) index of $h_{\zeta_{ij}}$.

The Riemannian metric $g = \text{Re}(\omega^{1,1})$ on X defines the L_2 -Riemannian metric on \mathcal{X}_{ij} :

$$(\delta_1 \phi, \delta_2 \phi) = \int_{\mathbb{R}} (\delta_1 \phi(t), \delta_2 \phi(t))_g dt,$$

and so we can speak about the gradient flow of $h_{\zeta_{ij}}$ on \mathcal{X}_{ij} . We will view a parametrized curve in \mathcal{X}_{ij} with parameter s as a map $\Phi = \Phi(t, s) : \mathbb{R}^2 \rightarrow X$ or, equivalently, as an X -valued function $\Phi(\tau)$ of one complex variable $\tau = t + \sqrt{-1}s \in \mathbb{C}$.

Definition 14.7. (a) The *Kähler gradient* of W (with respect to $\omega^{1,1}$) is the vector field $\text{grad}_{\omega^{1,1}}(W)$ on X (i.e., a smooth section of TX) obtained from the section dW of T^*X by applying the composite isomorphism

$$T^*X \xrightarrow{(\omega^{1,1})^{-1}} \overline{TX} \xrightarrow{v \rightarrow \bar{v}} TX,$$

where $(\omega^{1,1})^{-1}$ is the inverse to (14.1), and $v \rightarrow \bar{v}$ is the canonical \mathbb{C} -antilinear identification between \overline{TX} and TX .

(b) For $\zeta \in \mathbb{C}^*$, the ζ -instanton equation, or the *Witten equation* [FJR], is the following condition on an X -valued function $\Phi = \Phi(\tau) : \mathbb{C} \rightarrow X$:

$$\frac{\partial \Phi}{\partial \bar{\tau}} = -\bar{\zeta} \text{grad}_{\omega^{1,1}}(W).$$

To explain the meaning of the equation, at any $\tau_0 \in \mathbb{C}$

$$\frac{\partial}{\partial \bar{\tau}} \Phi(\tau_0) = \frac{\partial}{\partial t} \Phi(\tau_0) + \sqrt{-1} \frac{\partial}{\partial s} \Phi(\tau_0)$$

is a \mathbb{C} -linear combination of two tangent vectors to X at $\Phi(\tau_0)$, so itself an element of $T_{\Phi(\tau_0)}X$. It is required that this element is equal to the value of $-\bar{\zeta} \operatorname{grad}_{\omega^{1,1}}(W)$ at $\Phi(\tau_0)$.

The ζ -instanton equation can be seen as a complex $\bar{\partial}$ -analog of the ordinary differential equation of downward gradient flow for $\operatorname{Re}(\zeta W)$. In fact, it reduces to that equation, if we assume that $\Phi(t, s) = \phi(t)$ is independent on s .

Proposition 14.8 ([GMW]). *A curve $\Phi(s)$ in \mathcal{X}_{ij} represented by a map $\Phi = \Phi(t, s) : \mathbb{R}^2 \rightarrow \mathbb{C}$, is a gradient trajectory for $h_{\zeta_{ij}}$, if and only if it satisfies the ζ_{ij} -instanton equation. \square*

Examples 14.9. (a) If $X = \mathbb{C}$ with coordinate z and $\omega^{1,1} = dz d\bar{z}$ is the standard flat Kähler metric, then the equation has the form

$$\frac{\partial \Phi}{\partial \bar{\tau}} = -\bar{\zeta} \overline{W'(\Phi)}.$$

(b) More generally, if (z_1, \dots, z_k) is a local holomorphic coordinate system on X , and $\omega^{1,1} = \sum \omega_{a,\bar{b}}^{1,1} dz_a d\bar{z}_b$, then we can write $\Phi(\tau)$ as $(\Phi^1(\tau), \dots, \Phi^k(\tau))$ and the equation has the form

$$\frac{\partial \Phi^a}{\partial \bar{\tau}} = \left(\frac{\partial}{\partial t} + \sqrt{-1} \frac{\partial}{\partial s} \right) \Phi^a = -\bar{\zeta} \sum_b \eta^{a,\bar{b}} \frac{\partial \bar{W}}{\partial \bar{\Phi}^{\bar{b}}}.$$

Here $\|\eta^{a,\bar{b}}\|$ is the inverse matrix to $\|\omega_{a,\bar{b}}^{1,1}\|$.

E. The Maurer-Cartan element. Let $\mathfrak{g} = \mathfrak{g}_{A,N}$ be the L_∞ -algebra corresponding to our set A and the coefficient system given by the N_{ij} defined in §B. Thus, see §10,

$$\mathfrak{g} = \bigoplus_{Q'} N_{Q'} \otimes \operatorname{or}(\Sigma(A')), \quad A' = Q' \cap A,$$

where Q' runs over all convex subpolygons with vertices on A .

Fix one such Q' and suppose that it has vertices i_0, \dots, i_m (counterclockwise). Then the basis of $N_{Q'}$ is formed by the vectors e_ϕ for all gradient polygons $\phi = (\phi_{i_0 i_1}, \dots, \phi_{i_m i_0})$ that project (under W) onto the boundary $\partial Q'$. Let \mathfrak{w}' be the web dual to Q' , i.e., \mathfrak{w}' is the normal fan of Q' , see §13 and Fig. 28. Thus \mathfrak{w}' has one vertex and the sectors C_ν of \mathfrak{w}' are in bijection with the vertices i_ν of Q' .

Fix further a gradient polygon ϕ covering Q' and let $\mathcal{M}_\zeta(\phi)$ be the moduli space of solutions of the ζ -instanton equations $\Phi : \mathbb{C} \rightarrow X$ with the following asymptotic conditions:

- (1) When τ approaches infinity deep inside C_ν , then $\Phi(\tau)$ approaches x_{i_ν} .
- (2) Consider an edge \mathfrak{e} of \mathfrak{w}' separating some C_ν and $C_{\nu+1}$, see Fig. 28. For a point $\tau \in \mathfrak{e}$ let $l(\tau)$ be the line orthogonal to \mathfrak{e} passing through τ . Then, as $\tau \rightarrow \infty$ on \mathfrak{e} , the restriction $\Phi|_{l(\tau)}$ approaches the soliton $\phi_{i_\nu, i_{\nu+1}}$.

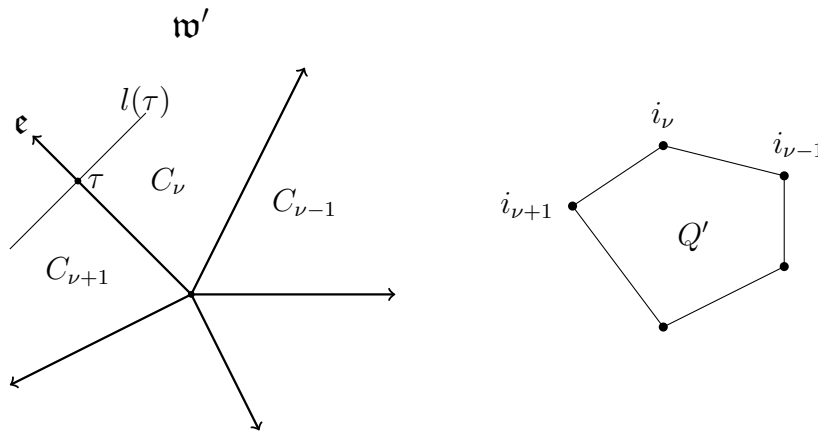


Figure 28: The asymptotic condition for ζ -instantons.

The proposal of [GMW] can be mathematically summarized as follows.

Conjecture 14.10. (a) Under sufficient genericity assumptions, $\mathcal{M}_\zeta(\phi)$ is a manifold of dimension $d(\phi) - 1$, equipped with a natural orientation.

(b) In particular, for ϕ such that $d(\phi) = 1$, i.e., e_ϕ is a basis vector of \mathfrak{g}^1 , we have a well defined *signed cardinality* $\gamma_\zeta(\phi) \in \mathbb{Z}$ of $\mathcal{M}_\zeta(\phi)$ (signs come

from the orientation of the 0-dimensional manifold). The element

$$\gamma = \gamma_\zeta = \sum_{d(\phi)=1} \gamma_\zeta(\phi) e_\phi \in \mathfrak{g}^1$$

is a Maurer-Cartan element.

(c) The deformation of the A_∞ -algebra R_∞ with respect to γ_ζ is identified with $\mathcal{RS}(\zeta W)$, the A_∞ -algebra associated to the Fukaya-Seidel category of ζW .

Remarks 14.11. (a) Since N_{ij} is spanned by the critical points of $h_{\zeta_{ij}}$ on \mathcal{X}_{ij} , it carries a natural Floer differential. This differential is in fact a part of the datum provided by α , corresponding to $m = 1$, i.e., to gradient polygons with 2 edges, so that $m = 1$ and $i_0 = i, i_1 = j$.

(b) Suppose that $m \geq 2$ and let ϕ be a gradient polygon lying over the boundary of a convex sub polygon Q' . In this case it seems likely that for any ζ -instanton $\Phi \in \mathcal{M}_\zeta(\phi)$, the composition

$$\mathbb{C} \xrightarrow{\Phi} X \xrightarrow{W} \mathbb{C}$$

maps \mathbb{C} bijectively onto the interior of Q' . If this is so, then one can think about ζ -instantons as “sections” of W over Q' .

(c) The argument justifying the Maurer-Cartan condition for γ proposed in [GMW] is based on “counting the ends” of the 1-dimensional manifolds $\mathcal{M}_\zeta(\phi)$ for $e_\phi \in \mathfrak{g}^2$. More generally, this type of argument suggests the existence of a compactification $\overline{\mathcal{M}}_\zeta(\phi)$ of $\mathcal{M}_\zeta(\phi)$ by adding strata which are products of $\mathcal{M}_\zeta(\psi)$ for “smaller” gradient polygons ψ . More precisely, strata are labelled by data \mathfrak{P} consisting of:

- (1) A regular polygonal subdivision $\mathcal{P} = \{Q''_\nu\}$ of the polygon Q' (image of ϕ under W).
- (2) An assignment, to each intermediate edge of the subdivision, of an instanton projecting onto this edge.

Such a datum can be thought of as a lifting of \mathcal{P} into X and is equivalent to a “colored web” in the terminology of [GMW]. Each polygon Q''_ν is then lifted to a gradient polygon ψ_ν , and the stratum $\mathcal{M}_\zeta(\mathfrak{P})$ is the product of the $\mathcal{M}_\zeta(\psi_\nu)$. In other words, the compactification $\overline{\mathcal{M}}_\zeta(\phi)$ has the strata

structure and the factorization property, completely parallel to those of the secondary polytope $\Sigma(A \cap Q')$.

Further, for a sequence of instantons (Φ_n) in $\mathcal{M}_\zeta(\phi)$, the condition of approaching the stratum $\mathcal{M}_\zeta(\mathfrak{P})$, i.e., the condition that $\lim \Phi_n \in \mathcal{M}_\zeta(\mathfrak{P})$, can be seen as a kind of tropical degeneration. That is, fixing $\varepsilon > 0$, consider the sets

$$U_n(\varepsilon) := \{\tau \in \mathbb{C} : \|d_\tau \Phi_n\|^2 \geq \varepsilon\}.$$

Then, as $\varepsilon \rightarrow 0$ and $n \rightarrow \infty$ simultaneously in a compatible way ($\forall \varepsilon \exists n$), the sets $U_n(\varepsilon)$ converge, in the Gromov-Hausdorff sense, to the web \mathfrak{w} dual to \mathcal{P} . The shape of each individual $U_n(\varepsilon)$ can be compared to the amoeba of a plane algebraic curve.

15 Speculations and directions for further work

1. From Stasheff polytopes to a categorical structure. A ζ_{ij} -instanton in §14 is the $m = 2$ case of the following “catastrophe” (non-generic occurrence) for a Morse function: a chain of m critical points of the same index connected by gradient trajectories in a sequential way. It was shown in [KS] that generic deformations of such a catastrophe are governed by the Stasheff polytope K_m whose vertices correspond to bracketings of $m + 1$ factors. As well known, combinatorics of the K_m is at the basis of the formalism of A_∞ -algebras and categories. It is therefore interesting to relate this appearance of the K_m to other categorical structures and to wall crossing formulas in 2 and more dimensions. This seems especially appealing since the K_m also describe [KS] higher syzygies among the Steinberg relations for elementary matrices $e_{ij}(\lambda) \in GL_N$:

$$e_{ij}(\lambda)e_{jk}(\mu) = e_{jk}(\mu)e_{ik}(\lambda + \mu)e_{ij}(\lambda).$$

Such relations, each having the shape of a pentagon K_3 , appear naturally in the wall-crossing formulas.

2. Higher TFT structures and E_d -algebras. It seems certain that in d dimensions the L_∞ -algebra \mathfrak{g} from §3 can be refined to an E_d -algebra, and \mathfrak{g}_∞ to an E_{d-1} -algebra. We plan to address this in a subsequent paper, currently in preparation. The possibility of such refinement is in agreement with the higher analog of the Deligne conjecture [KoSo2, Lur, Gin] which says that the deformation complex of an E_{d-1} -algebra is naturally an E_d -algebra. The quasi-isomorphism Φ in the Universality Theorem should then be a morphism of E_d -algebras. Further, the full higher-dimensional analog of the concept of a coefficient system from §10 should associate to any simplex of codimension p with vertices in A , an E_p -algebra.

3. Curvilinear theory. It would be interesting to develop the analog of the formalism of Fukaya-Seidel categories and of [GMW] for the case when the potential W takes values not in \mathbb{C} but in some Riemann surface S . In this case the set A of critical values makes S into a “marked surface” in the sense of Teichmüller theory and one can use isotopy classes of curvilinear triangulations of S with vertices in A in order to analyze various categorical structures. In particular, geodesics of quadratic differentials as well as spectral networks of Gaiotto-Moore-Neitzke [GMN] seem like natural objects to appear in such a theory.

4. Infinite-dimensional case. Much of the recent “physical” work on Picard-Lefschetz theory and its generalizations was motivated by the infinite-dimensional example of the complexified Chern-Simons functional [Wit]. Here X is the (universal cover of the) space of all connections in a principal bundle on a 3-manifold M , whose structure group is a complex semi simple Lie group. The set A of critical values of the Chern-Simons functional consists of complex numbers of substantial arithmetic importance (involving volumes of hyperbolic manifolds, regulators of elements of $K_3(\overline{\mathbb{Q}})$ and the like). The nature of polygons and triangulations that can be formed out of these numbers remains mysterious. It would be very interesting if Picard-Lefschetz theory imposed some constraints on the convex geometry of these numbers.

The case of the holomorphic Chern-Simons functional (M is a 3-dimensional complex Calabi-Yau) is even more mysterious.

References

- [BGS] L. J. Billera, I.M. Gelfand, B. Sturmfels. Duality and minors of secondary polyhedra, *Journal of Combinatorial Theory, Series B*, **57** (1993), 258-268.
- [BS] L. J. Billera, B. Sturmfels. Fiber polytopes. *Ann. of Math.* **135** (1992), 527-549.
- [BK] A.I. Bondal, M.M. Kapranov. Representable functors, Serre functors and mutations. *Math. USSR Izv.* **35** (1990), 519-541.
- [CGw] K. Costello, O. Gwylliam. Factorization algebras in quantum field theory. Book draft, available at www.math.northwestern.edu/~costello/factorization.pdf.
- [Cur] J. Curry. Sheaves, Cosheaves and Applications. arXiv:1303.3255.
- [FJR] H. Fan, T. Jarvis, Y. Ruan. The Witten equation, mirror symmetry and quantum singularity theory. *Ann. Math.* **178** (2013), 1-106.
- [GiKa] V. Ginzburg, M.Kapranov, Koszul duality for operads, *Duke Math. J.* **76** (1994) 203-272. .
- [GMN] D. Gaiotto, G. W. Moore, A. Neitzke. Spectral networks. *Ann. Henri Poincaré*, **14** (2013) 1643- 1731.
- [GMW] D. Gaiotto, G. Moore, E. Witten. Algebra of the infrared: string field theoretic structures in massive $\mathcal{N} = (2, 2)$ field theory in two dimensions. In preparation.
- [GKZ] I.M. Gelfand, M.M. Kapranov, A.V. Zelevinsky. Discriminants, Resultants and Multidimensional Determinants. Birkhauser, Boston, 1994.
- [GM] S. Gelfand, R. D. MacPherson. Verma modules and Schubert cells: a dictionary. in: “*Paul Dubreil and Marie-Paule Malliavin Algebra Seminar*”, *34th Year (Paris, 1981)*, pp. 1-50. Lecture Notes in Math. **924**, Springer, Berlin-New York, 1982.
- [Get] E. Getzler. Lie theory for nilpotent L_∞ -algebras. *Ann. of Math.* **170** (2009), 271-301.
- [Gin] G. Ginot, Notes on factorization algebras, factorization homology and applications. arXiv:1307.5213.

- [HIV] K. Hori, A. Iqbal, C. Vafa. D-branes and mirror symmetry. arXiv:hep-th/0005247.
- [KS] M. Kapranov, M. Saito. Hidden Stasheff polytopes in algebraic K-theory and in the space of Morse functions. In: “Higher homotopy structures in topology and mathematical physics” (Poughkeepsie, NY, 1996), 191-225, *Contemp. Math.*, 227, Amer. Math. Soc., Providence, RI, 1999.
- [Ko1] M. Kontsevich, Operads and Motives in Deformation Quantization, arXiv:math/9904055.
- [KoSo1] M. Kontsevich, Y. Soibelman, Deformation theory I, Book draft, 2007, available at www.math.ksu.edu/~soibel.
- [KoSo2] M. Kontsevich, Y. Soibelman, Deformation of algebras over operads and Deligne’s conjecture, arXiv:math/0001151.
- [KoSo3] M. Kontsevich, Y. Soibelman, Homological Mirror Symmetry and torus fibrations, arXiv:math/0011041.
- [Lur] J. Lurie. Derived Algebraic Geometry VI: E_k -algebras. arXiv:0911.0018.
- [OM] A. Björner, M. Las Vergnas, B. Sturmfels, N. White, G. Ziegler. Oriented Matroids. *Encyclopedia of Mathematics and Its Applications* 46 (2nd ed.). Cambridge University Press, 1999.
- [Se1] P. Seidel, Fukaya categories and Picard-Lefschetz theory, European Mathematical Society, Zurich, 2008.
- [TV] B. Toën, M. Vaquié. Moduli of objects in dg-categories. *Ann. Sci. École Norm. Sup. (4)* **40** (2007), 387-444.
- [Wit] E. Witten, Analytic Continuation Of Chern-Simons Theory, arXiv:1001.2933.

M.Ka.: Kavli Institute for Physics and Mathematics of the Universe (WPI), 5-1-5 Kashiwanoha, Kashiwa-shi, Chiba, 277-8583, Japan. Email: mikhail.kapranov@ipmu.jp

M.Ko.: Institut des Hautes Études Scientifiques, 35 Route de Chartres, 91440 Bures-sur-Yvette, France. Email: maxim@ihes.fr

Y. So.: Department of Mathematics, Kansas State University, Cardwell Hall, Manhattan, KS 66506 USA. Email: soibel@math.ksu.edu