A Conjecture on Maurer-Cartan Element

— Lecture on Fomenko's Seminar,
Moscow State University

Huijun Fan. Peking University

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Content:

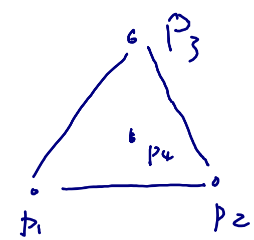
1) Second polytope $\Sigma(A)$ of a set $A \subset \mathbb{R}^d$, $d \geq 1$
2) L^∞- structure $\mathfrak{g}$ induced by $\Sigma(A)$
3) Maurer-Cartan equation of $\mathfrak{g}$.
4) $d=1$ Case
5) Coefficient system $N = (N_\delta)$ and extended L^∞-structure $\mathfrak{g}_{N,\infty}$.
6) $d=2$ Case and universality theorem.
7) Landau-Ginzburg model and MCE Conjecture
8) Fukaya Category of LG model

Lecture Π based on 3 papers:

A) Gaiotto-Moore-Witten: algebra of the infrared
B) Kapranov-Kontsevich-Soibelman: algebra of the infrared and secondary polytopes (arxiv.1812.11748)
c) H. Fan, W. Jiang and D. Yang. Fukaya Cat. of LG model (2018)

1° Second polytope $\Sigma(A)$ of a finite set $A \subset \mathbb{R}^d$ ②

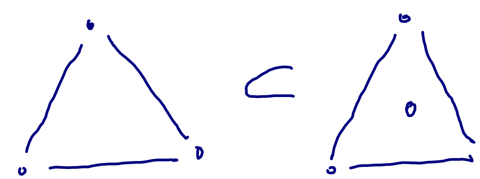
- $A \subseteq \mathbb{R}^d$ is a finite set of points; all points are in general position. $Q := \mathrm{Conv}(A)$.



$d = 2$
$|A| = 4$

- (Q, A) : marked polytope

- marked subpolytope $(Q', A') \subset (Q, A)$, if (Q', A') is a marked polytope and $A' \subset A$ (allow $A' \subset\subset A$)



- <u>Def</u> polyhedral subdivison $\mathcal{P} = \{(Q_v, A_v)\}$ s.t.

1) $(Q_v, A_v) \subset (Q, A)$, $\dim Q_v = \dim Q$

2) $Q = \cup\, Q_v$

3) $Q_v \cap Q_{v'}$ is the common face of

$$A_v \cap (Q_v \cap Q_{v'}) = A_{v'} \cap (Q_v \cap Q_{v'})$$

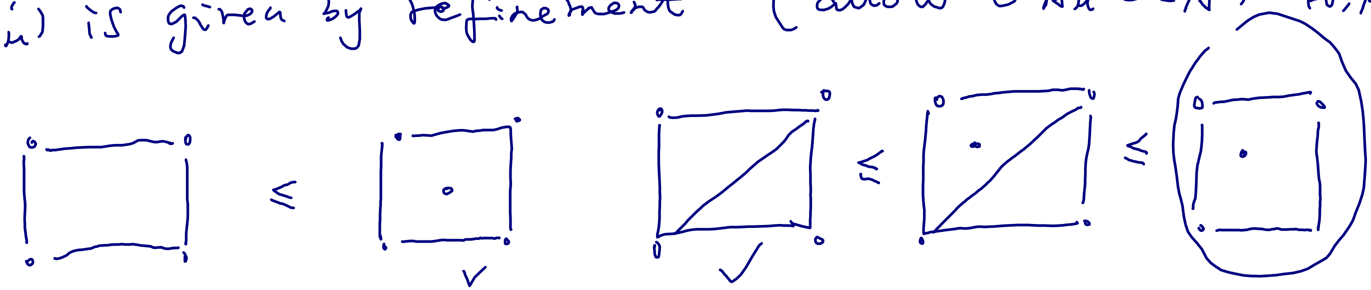
③

$\triangle$ All (regular) polyhedral subdivision $\mathcal{P}$ forms a poset

$R(Q,A)$, the partial order $\mathcal{P}' = \{(Q'_\mu, A'_\mu)\} \leq \mathcal{P} = \{(Q_\nu, A_\nu)\}$

if (Q'_μ, A'_μ) is given by refinement (allow $\cup A'_\mu \subset A$) (Q,A)

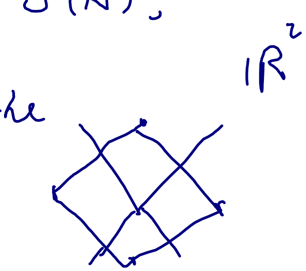
Ex :



$\triangle$ $R(Q,A)$ can be identified with the secondary fan $S(A)$;

the dual polyhedral $\Sigma(A)$ of $S(A)$ is called the

second polytope $\Sigma(A)$.



$\triangle$ There are different ways to construct $S(A)$ and $\Sigma(A)$

choose one

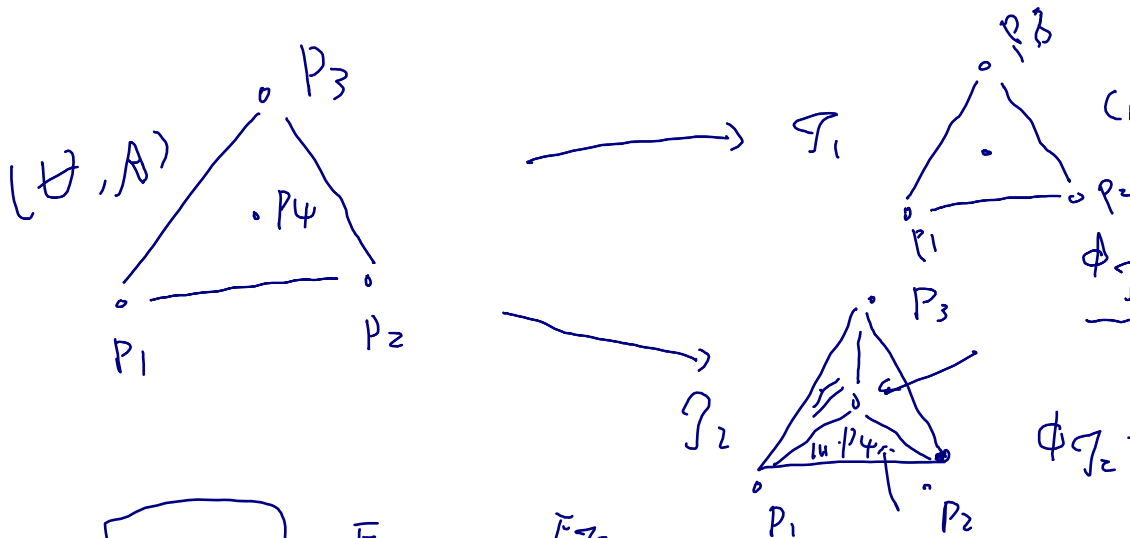
④ Construction of $\Sigma(A)$: Take a transition inv. volume in $\mathbb{R}^d$;

Let $\mathcal{T} \in R(\theta, A)$ be a regular triangulation, then $\underline{\phi_{\mathcal{T}}} \in \mathbb{R}^A$ is given by

$$: \forall \omega \in A, \quad \underbrace{\phi_{\mathcal{T}}(\omega) := \sum_{\substack{\Delta \in \mathcal{T} \\ \omega \in \mathrm{Ver}(\Delta)}} \mathrm{vol}(\Delta)}$$

Then $\Sigma(A) := \mathrm{Conv}(\phi_{\mathcal{T}},\ \mathcal{T} \in R(\theta, A)) \subset \mathbb{R}^d$

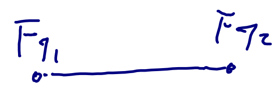
Ex



(no division)

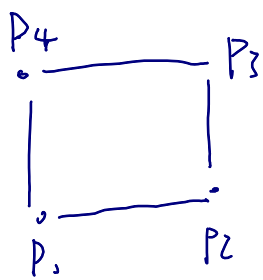
$\underline{\phi_{\mathcal{T}_1}} = (\underline{|\Delta|}, \underline{|\Delta|}, \underline{|\Delta|}, 0)$

$\phi_{\mathcal{T}_2} = \left(\frac{2}{3}|\Delta|, \frac{2}{3}|\Delta|, \frac{2}{3}|\Delta|, \underline{|\Delta|}\right)$

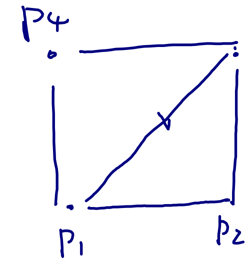
So $\boxed{\Sigma(A)} =$ 

$F_{\mathcal{T}_1}$ is a vertex of $\Sigma(A)$

(5) **Ex**

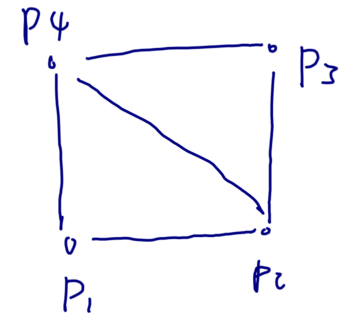


$(\theta, A) \longrightarrow T_1$



$\phi_{T_1} = (1, \frac{1}{2}, 1, \frac{1}{2})$

$\longrightarrow T_2$



$\phi_{T_2} = (\frac{1}{2}, 1, \frac{1}{2}, 1)$

So $\Sigma(A) = \overset{F_{T_1}}{\bullet}\!\!-\!\!\!-\!\!\!-\!\!\!-\!\!\overset{F_{T_2}}{\circ}$ (line in $\mathbb{R}^4$)

$A \subset \mathbb{R}^d$

Prop

$\mathcal{R}(\theta, A) \longleftrightarrow \Sigma(A)$

$\mathcal{P} \longleftrightarrow F_{\mathcal{P}}$ (face)

$\mathcal{P}' \le \mathcal{P} \longleftrightarrow F_{\mathcal{P}'} \le F_{\mathcal{P}}$

a) $\dim \Sigma(A) = |A| - d - 1$; b) Vertex of $\Sigma(A) \longleftrightarrow$ triangulation

c) edge of $\Sigma(A) \longleftrightarrow$ flips in triangulation

⑥ d) Codim-1 face F in $\Sigma(A)$ $\longleftrightarrow$ Coarse Subdivision
(GKZ languages)
(GMW: taut webs)

Two important Cases:

i) Assume $w \in A - \mathrm{Ver}(Q)$, then $(Q, A - \{w\}) \preceq (Q, A)$

ii) $\exists$ hypersurface H in $\mathbb{R}^d$ separates Q into 2 subpolytopes Q_1, Q_2, $A_\nu := A \cap Q_\nu$ then

$$\{(Q_1, A_1), (Q_2, A_2)\} \preceq (Q, A)$$

Thm (Factorization) $\mathcal{P} \in R(Q, A)$, $\Rightarrow$ face $F_{\mathcal{P}} \subset \Sigma(A)$.
Let $\mathcal{P} = \{(Q_\nu, A_\nu)\}$, then $F_{\mathcal{P}} = \prod_\nu \Sigma(A_\nu)$

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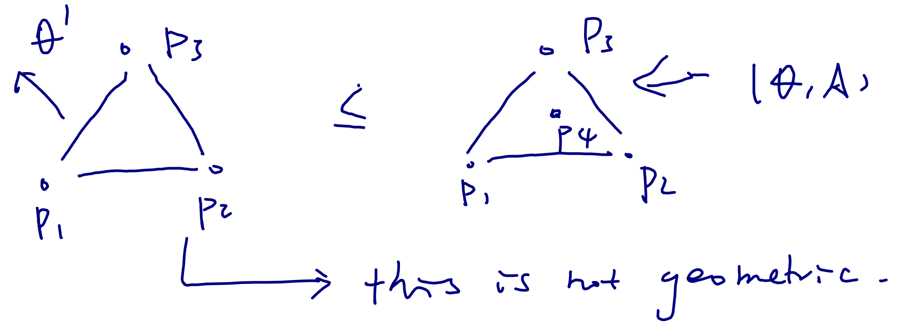
Def geometrical marked subpolytope is $(Q', A') \subset (Q, A)$

if $A' = A \cap Q'$



$A' = \{p_1, p_2, p_3\}$ $\quad$ $A = \{p_1 \cdots p_4\}$

2) L_∞-alg. $\mathfrak{g}$ induced by $\Sigma(A)$

▷ Let $\underline{P}$ be a convex polyhedron $\Rightarrow$ cellular chain complex $C_\bullet(P)$ (dg-space)

$$C_\bullet(P) = \bigoplus_{\substack{F \subset P \\ \uparrow \\ \text{face}}} \underline{or(F)}\,\underline{[\dim F]}$$

$$\underline{or(F) = H_c^{\dim F}(\mathring{F}, \mathbb{R})}$$
(1-dim. oriented space)

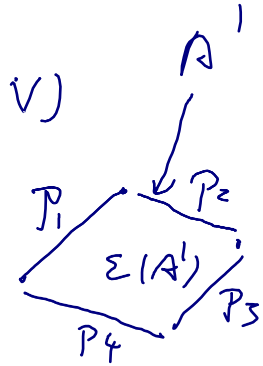
▷ Now take $\underline{P = \Sigma(A)}$

- graded vec. space $\underline{V} = \bigoplus_{(\theta', A') \subset (\theta, A)} V_{A'}$ $\qquad A' \subset A$, $\underline{[w] \text{ s.t.}} \to \Sigma(A')$

$$V_{A'} = \underline{or(\Sigma(A'))}\,[\dim \Sigma(A')] \qquad \bigoplus \underbrace{V \odot \cdots \odot V}\ = S^\bullet(V)$$

$\Rightarrow$ symm alg. $\underline{S^\bullet(V)}$ given by grad. comm. multi. $\underline{\odot}$

9 • degree 1 differential $d : S^\bullet(V) \longmapsto S^\bullet(V)$



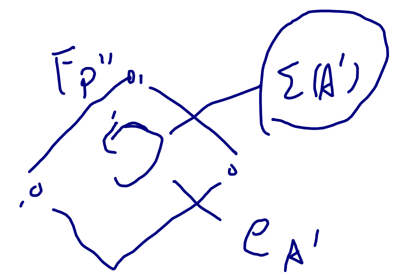
$\langle e_{A'} \rangle = V_{A'} = \mathrm{or}(\Sigma(A'))[\dim(\Sigma(A'))]$

$\partial\Delta = (+1)\overline{p_2 p_3} + (-1)\, p_3 p_1 + p_1 p_2$

Each face $F_{\mathcal{P}''} \longleftrightarrow$ coarse polyhedral subdivision $\mathcal{P}''$

$= \{(Q''_\nu, A''_\nu)\}$

This face defines a 1-dim subspace



$$V_{\mathcal{P}''} = \bigotimes_\nu V_{A''_\nu} \subset S^\bullet(V)$$

$$d(e_{A'}) := \sum_{\mathcal{P}''\text{ coarse}} d_{\mathcal{P}''}(e_{A'}), \quad d_{\mathcal{P}''} : V_{A'} \longmapsto V_{\mathcal{P}''}$$

coefficient is given by the top diff in $C_\bullet(\Sigma(A'))$

10 **Prop.** $(S^\bullet(V), d)$ is a comm. dg-algebra. ($d^2=0$)

(factorization of $\Sigma(A)$)

$\Leftarrow$ duality

Thm (Kontsevich-Soibelman)

Let $\mathfrak{g}$ be a fin-dim dg Lie alg. then its Chevalley-Eilenberg cochain complex is a comm. dg-alg:

$$\boxed{C^\bullet_{Lie}(\mathfrak{g})} = S^\bullet(\mathfrak{g}^*[-1])$$

$S^\bullet(V) \overset{d}{=} \supset S_+ = \bigoplus_{n\geq 1} V^n$

A general alg diff d on the ideal $S^\bullet_+(\mathfrak{g}^*[-1])$ will induce a L_∞-structure on $\mathfrak{g}$.

$\overset{\bullet}{\mathfrak{g}}$ L_∞-algebra.

Apply to $(S^\bullet(V), d) \Rightarrow \overset{\bullet}{\mathfrak{g}} = \overset{\bullet}{\mathfrak{g}}_A = V^*[-1] = \bigoplus_{(\Theta', A') \subset (\Theta, A)} E_{A'}$

$E_{A'} = V^*_{A'}[-1] = or(\Sigma(A'))^*[-1-\dim(\Sigma(A'))]$ has a L_∞-structure $\{\lambda_n\}$

(11)

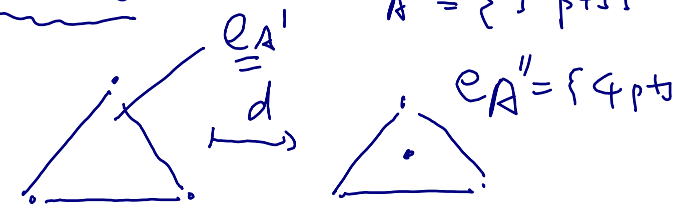
$$\mathfrak{g} = \mathfrak{g}_A = \bigoplus_{\substack{(Q',A') \\ \text{geom}}} E_{A'} = \bigoplus_{Q' \subset (Q,A)} E_{A \cap Q'} \subset \mathring{\mathfrak{g}}$$

<u>prop</u> $\mathfrak{g}_A$ is an L_∞-subalg in $\mathring{\mathfrak{g}}_A$ with <u>trivial differential.</u>

$\mathring{\mathfrak{g}}$

$\triangle$ Geometrical meaning of $\lambda_n : \mathring{\mathfrak{g}}^{\otimes n} \longmapsto \mathring{\mathfrak{g}}$ $\quad A' = \{3 \text{ pts}\}$

$\lambda_1 = d$ (= add one marked pt)



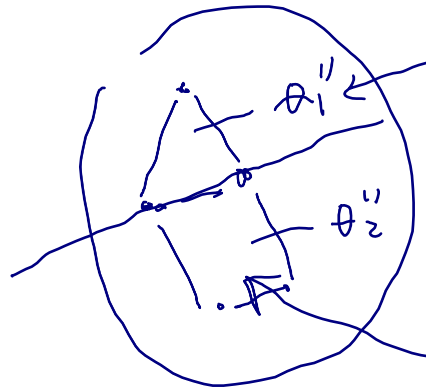
$\underset{=}{\lambda_n}$ Corresponds to: (Q', A') when do Coarse Subdivison

get exactly n marked subpolytope $\{(Q''_i, A''_i)\}$

$\Rightarrow$ $\underline{\langle e_{A''_1} \otimes \cdots \otimes e_{A''_n} \mid \lambda_n \mid e_{A'} \rangle = \pm 1}$ $\qquad \lambda_n(e_{A''_1} \otimes \cdots \otimes e_{A''_n})$

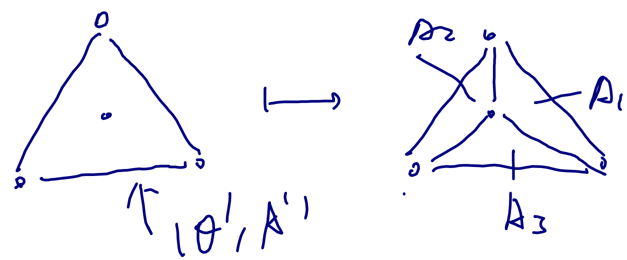
$= \sum (\cdots) e_{A'}$

(12)



$(\theta', A') = (\theta_1'', A_1'') \cup (\theta_2'', A_2'')$

$\lambda_2(e_{A_1''}, e_{A_2''}) = [e_{A_1''}, e_{A_2''}] = \pm e_{A'} \to \theta'$



$\lambda_3(e_{A_1} \otimes e_{A_2} \otimes e_{A_3}) = e_A$

base: $\quad g^{\leq 0} = 0;$ $\quad g^1$: spanned by $e_{A'}$, where

$|A'| = d+1$ (d-dim simplex)

g^2: spanned by circuit $A' \subset A$ □

$|A'| = d+2$

3) Maurer-Cartan equation

$(\mathfrak{g}, \lambda_n)_{n \geq 1}$

$\searrow$ L_∞-alg.

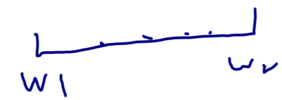
$$\underline{\sum_{n=0}^{\infty} \frac{1}{n!} \lambda_n(\gamma, \ldots, \gamma) = 0 \in \mathfrak{g}^2.}$$

$\underline{\underline{\gamma}} = \sum \underline{\underline{\gamma_\delta}} \cdot e_\delta \in \mathfrak{g}^1$

$\hookrightarrow$ d-dim simplex.

- $\underline{MC(\mathfrak{g})} \subset \mathfrak{g}^1$ is an <u>affine scheme. describes</u> the deformation of $\mathfrak{g}$.

4) $d=1$ Case (L_∞ alg to A_∞ alg)



$A = \{w_1 < w_2 < \cdots < w_r\}$. $\quad Q = \mathrm{Conv}(A) = [w_1, w_r]$

Let (Q', A') be a marked subpolyhedral, $\underline{P}'' = \{(Q''_\nu, A''_\nu)\}$

Since Q''_ν has a natural order. $\Rightarrow$ $S^\circ(V)$ Can be lifted to the tensor algebra $T^\circ(V)$ as follows: $\odot \;\; \underline{\otimes}$

$$P'' \in R(Q, A) \Rightarrow V_{P''} \triangleq \bigotimes_\nu V_{A''_\nu}$$

$\langle e_{A'} \rangle = V_{A'}$ $\quad d(e_{A'}) = \displaystyle\sum_{P''\text{ coarse}} d_{P''}(e_{A'})$; $\quad d_{P''} : V_{A'} \mapsto V_{P''}$

is defined by the chain differential in $C_\circ(\Sigma(A'))$

$\Rightarrow$ $(T^\circ(V), d)$ is an associative-dg alg.

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Since d preserves the ideal $T_+(V) = \bigoplus_{n\geq 1} V^{\otimes n}$

Take the dual, the L_∞-structure of $\mathring{g}$ can be lifted to

a A_∞-stru. $\mathring{R}$. s.t. $[x, y] = xy - yx$.

Similarly for g, we have the lift R.

Note that the base of R is given by e_{ij}, $1 \leq i < j \leq r$

corresponds to $([w_i, w_j], \{w_i, \dots, w_j\}) = \mathfrak{X}$

$\hookrightarrow$ $j-i+1$ pts

degree $|e_{ij}| = j - i$. $\boxed{m_1 = 0}$ $|\Sigma(\mathfrak{X})| = j-i+1$

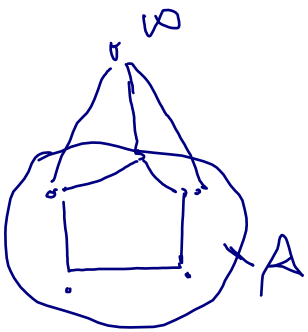
m_2 is given by $\boxed{e_{ij} \cdot e_{jk} = e_{ik}}$ $(i<j<k)$ w_i w_j w_k

$\Rightarrow$ R is an assoc. alg without unit $\cong$ strictly upper triangular $(r \times r)$ matrix.

(16) * Add pt at ∞.

$A \subset \mathbb{R}^d \Rightarrow (Q, A)$ marked polytope. $\tilde{A} = A \cup \{\infty\}$ (in gene. position)

$\tilde{Q} = \mathrm{Conv}(\tilde{A})$; since the subpolyhedrals $(Q', A') \subset (\tilde{Q}, \tilde{A})$



are classified as finite polytope and infinite polytope.

$\Sigma(\tilde{A}) \longleftrightarrow$

$\Rightarrow L_\infty$-alg $\tilde{g}$ splits $\tilde{g} = g_A \oplus g_\infty$

$E_{A'} \longleftrightarrow \Sigma(A')$

where g_A is gen. by $E_{A'} = or(\Sigma(Q' \cap A))^* \cdot [-1-\dim \Sigma(Q' \cap A)]$

where $Q' = \mathrm{Conv}(A')$ is finite

prop. g, g_∞ are L_∞-subalgebra, g_∞ is L_∞-ideal.

$\underbrace{\lambda_k(l_1 \cdots l_k)}_{\in g_\infty} \in g_\infty$

(17) General Case

$\underline{X}$: CW complex (X_α) $\Rightarrow$ Cellular Cochain Complex

$$\underbrace{\bigoplus_{\dim X_\alpha = 0} or(X_\alpha) \longmapsto \bigoplus_{\dim X_\alpha = 1} or(X_\alpha) \longmapsto \cdots}$$

Take a sheaf $\underline{F}$ of k-vec. Spaces on $\underline{X}$ which is Constant along each $\underline{X_\alpha}$. Let $\underline{F_\alpha} = H^0(X_\alpha, \underline{F})$ (stalk of F)

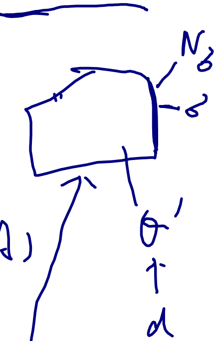
then $\underline{\gamma_{\alpha\beta}} : \underline{F_\alpha} \longmapsto \underline{F_\beta}$ if $\alpha \leq \beta$, $(\underline{X_\alpha} \subset \overline{X_\beta})$

Coeff sys

$$\underline{C^\circ(X, F)} := \{ \underline{\bigoplus_{\dim X_\alpha = 0} or(X_\alpha) \otimes F_\alpha} \longmapsto \bigoplus_{\dim X_\beta = 1} or(X_\beta) \otimes \underline{F_\beta} \longmapsto \cdots \}$$

(18) 5) Coefficient system $N = (N_\sigma)$ and extended L_∞-stru. $\mathfrak{g}_{N,\infty}$

Def. The Coeff. system $N = (N_\sigma)$ Consists of a family of cochain complex N_σ where σ is a (oriented) $(d-1)$-Simplex, $\mathrm{Vert}(\sigma) \subset A$, and satisfies $N_{\bar\sigma} = N_\sigma^*$



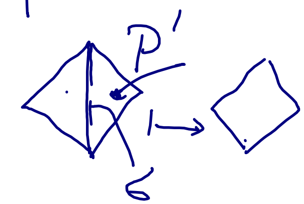
Define $N_{A'} = \bigotimes_{\sigma \subset \partial\Delta'} N_\sigma$, where $(\Delta', A') \subset (\Delta, A)$

* If $\mathcal{P} = \{(\Delta_v', A_v')\}$ is a subdivision of (Δ, A).

$N_{A_v'} \hookrightarrow \Sigma(A_v')$

then $N_{\mathcal{P}} := \bigotimes_v N_{A_v'}$

* If $\mathcal{P}' < \mathcal{P}$ is a refinement, then $\exists$ homomorphism of cochain complexes: $\gamma_{\mathcal{P}'\mathcal{P}} : N_{\mathcal{P}'} \longmapsto N_{\mathcal{P}}$ which is given by taking the trace $N_\sigma \otimes N_{\bar\sigma} \longmapsto \mathbb{R}$



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prop. $\mathcal{P}'' < \mathcal{P}' < \mathcal{P}$, then $\gamma_{\mathcal{P}''\mathcal{P}} = \gamma_{\mathcal{P}'\mathcal{P}} \circ \gamma_{\mathcal{P}''\mathcal{P}'}$

Now given a coeff. system $(N_{\mathcal{P}}, \gamma_{\mathcal{P}'\mathcal{P}})$

coarse subdiv.

$\Rightarrow$ sheaf $\mathcal{N}_A$ over $\Sigma(A)$; since $F_{\mathcal{P}}^{\vee} = \prod_{\nu} \Sigma(A'_\nu)$

$$\mathcal{N}_A\big|_{F_{\mathcal{P}}} = \bigotimes_{\nu} \mathcal{N}_{A'_\nu}$$

$\mathfrak{g}_N$ ↑

$\triangle$ $\quad V_N^{coh} = \bigoplus_{\substack{A' \subset A \\ |A'| \geq d+1}} V_{A'}^{coh}$ $\qquad V_{A'}^{coh} = \mathcal{N}_{A'} \otimes or(\Sigma(A'))[\dim(\Sigma(A'))]$

$\Rightarrow$ $(S^{\cdot}(V_N^{coh}), d)$ is comm. dg-alg.

prop: a) The cochain differential of $\mathcal{N}_{A'}$ gives the L_∞-alg str.

$$\mathring{\mathfrak{g}}_N = \mathring{\mathfrak{g}}_{A,N} = \bigoplus_{(Q', A') \subset (Q, A)} E_{A'}$$

(2)

$$E_{A'} = \overline{N}_{A'} \otimes or(\Sigma(A'))^* [-\dim \Sigma(A') - 1]$$

L_∞-alg $\mathring{g}_N$ is nilpotent.

b) $g_N = g_{A-N} = \bigoplus_{A' \subset (A,A)} E_{A' \cap A}$ is L_∞-subalg.

$$MC(g_N) \subset g_N^! = \bigoplus_{A' \subset (A,A)} N_{A'}^{-\dim \Sigma(A')} \otimes or(\Sigma(A'))^*, \quad A^! = A \cap A'$$

(21) * Relative Case (add ∞)

$A \Rightarrow \underline{\tilde{A} = A \cup \{\infty\}}$. We have

Coeff. $N \Rightarrow \tilde{N}$ $\qquad$ $\mathfrak{g}_{\tilde{N}} = \mathfrak{g}_N \ltimes \mathfrak{g}_{\tilde{N},\infty}$

$\mathfrak{g}_{\tilde{N},\infty}$ is the direct sum of summands $E_{A'}$ corresp.

to inf. marked subpolytopes (Q', A'). $\swarrow$ duality

$\Longrightarrow$ $\left(S^{\bullet}(\mathfrak{g}^{*}_{\tilde{N}}[-1]), d\right) = \left(S^{\bullet}((\mathfrak{g}_N \oplus \mathfrak{g}_{\tilde{N},\infty})^{*}[-1]\right)$

Is a Comm. dg-algebra. by KS theorem, this is equivalent

to 1) A L_∞-alg $\mathfrak{g}_N$; 2) A L_∞-alg $\mathfrak{g}_{\tilde{N},\infty}$

3) A L_∞-morphism $\Psi : \mathfrak{g}_{\tilde{N}} \longrightarrow R\,Der_{Lie}(\mathfrak{g}_{\tilde{N},\infty})$ $\quad \mathfrak{g}_\infty$

L_∞-alg $\qquad$ Eilenberg-Chevalley cochain cpl.

$A \subset \mathbb{R}^d$

$\tilde{A} = A \cup \{\infty\}$

(Q, A), $(\tilde{Q}, \tilde{A})$

$\downarrow \qquad \downarrow$

$\mathfrak{g} \qquad \tilde{\mathfrak{g}} = \mathfrak{g} \oplus \mathfrak{g}_\infty$

$\Downarrow$

$(S^{\bullet}(\tilde{\mathfrak{g}}^{*}[-1]), d)$

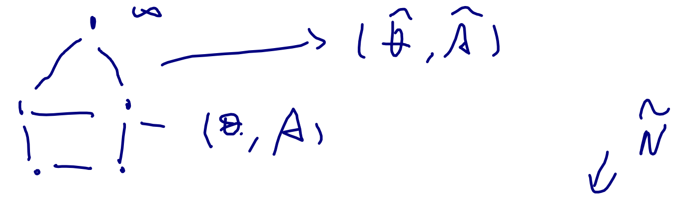
Comm. dgs

als

$\boxed{dg\,(A}$

(22) 6) $\boxed{d=2 \quad \text{Case}}$

In this case $\mathfrak{g}_{\tilde{N}-\infty}$ can be lifted to A_∞-alg. $R_{\tilde{N}}$ ∞



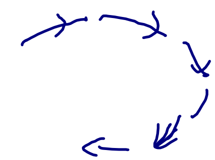
and we have L_∞-morphism

A_∞-alg

$$\Psi : \mathfrak{g}_{\tilde{N}} \longmapsto (R.\mathrm{Der}_{A_\infty}(R_{\tilde{N}-\infty}))$$

$$= C^{\geq 1}(R_{\tilde{N},\infty}, R_{\hat{N},\infty})[1]$$

(truncated Hochschild cochain cpl)



DGLA

$C^\bullet(R, R)$

$C^n(R,R) = \mathrm{Hom}_k(R^{\otimes n}, R)$, d, $\cup$, $[\phi, \psi]$

L_∞-alg $\mathfrak{g}_N$

$\lambda_1 = 0$

there $\exists$ a dg Lie-algebra structure

Thm (universality, KKS) $\boxed{d=2}$. $\Psi : \mathfrak{g}_N \mapsto C^{\geq 1}(R_N, R_N)[1] \to d \neq 0$

factors through $\Phi : \mathfrak{g}_N \mapsto \vec{C}^{\geq 1}(R_N, R_N)[1]$, and Φ is quasi-isom.

directed Hoch-cochain cpl.

(23) 7) Landau-Ginzburg model and MCE Conjecture

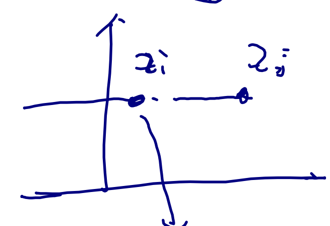
(X, W) LG model $\Longleftrightarrow$ X noncpt complete Kähler mfd ω (Kähler) form.

$W : X \mapsto \mathbb{C}$ holo ftn (assume to be Morse)

$|V| := \mathrm{Crt}(W) = \{x_1, \dots, x_r\}$, $A = W(|V|)$ is the set of crit values $\subset \mathbb{R}^2 \cong \mathbb{C}$

$\Rightarrow$ $(\theta, A) \Rightarrow$ secondary polytope $\Sigma(A)$. Now we put a $\mathbb{C}$ $\rightsquigarrow$ S S_∞

Coefficient system (S_i, N_{ij}) on A: $S_i = \mathbb{R}$ on x_i



$N_{ij} = \{\phi_{ij}\}$, where ϕ_{ij} is the gradient flow line $T_S(x_i) \cap T_U(x_j) \neq \emptyset$

from x_i $(t \to -\infty)$ to x_j $(t \to \infty)$ for $\mathrm{Re}(e^{\sqrt{-1}\theta_{ij}} W)$, where

$$e^{\sqrt{-1}\theta_{ij}} = \gamma_{ij} = \left(\frac{W_i - W_j}{|W_i - W_j|}\right)^{-1}$$

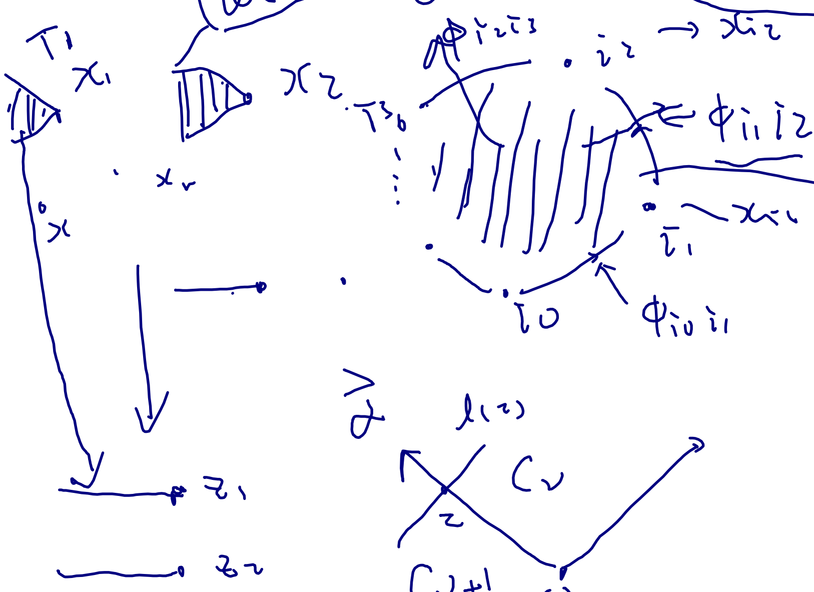
$\dim N_{ij} = n_{ij} =$ Intersection # of $T_i^{\theta_{ij}}$ and $T_j^{\theta_{ij}}$ $\boxed{\mathrm{Re}(W \cdot \xi)}$

24

ReW

Lefschetz

gradient $\boxed{\text{polygon}}$



$\mathrm{Re}(\zeta W)$

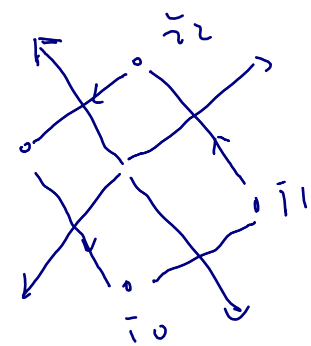
$\vec{\Phi} = (\phi_{i_0 i_1}, \phi_{i_1 i_2} \cdots \phi_{i_m i_0}) \xrightarrow{w} \mathfrak{g}'$

(cyclic fan of solitons)

interior part

$\mathcal{M}_{\zeta}(\vec{\phi}) = \{$ sol. of $\boxed{\zeta W}$ Witten equ $\} \Rightarrow \Phi : \mathbb{C} \longmapsto X$.

← potential



s.t. as $z \mapsto \infty$; $\Phi(\ell(z))$

$\longmapsto \phi_{\nu\nu+1}$

GMS

Moslov index

1° $\mathcal{M}_\zeta(\vec{\phi})$ is a oriented mfd of dim. $d(\vec{\phi}) - 1$ $\quad \vec{\phi}$

2° for $\vec{\Phi}$ s.t. $d(\vec{\phi}) = 1$, i.e. $e_{\vec{\phi}} \in \mathfrak{g}'$ take $C_{\mathfrak{g}}$ $\quad e_{\vec{\phi}}$

$\boxed{\# \mathcal{M}_\zeta(\vec{\phi})} = \boxed{\gamma_\zeta(\vec{\phi})} \in \mathbb{Z}$, then

$\gamma = \sum_{d(\vec{\phi})=1} \gamma_\zeta(\vec{\phi}) e_{\vec{\phi}} \in \mathfrak{g}'$ is a MCE.

MCE Conj:

$\Phi : \mathbb{C} \longmapsto X$

Witten equation

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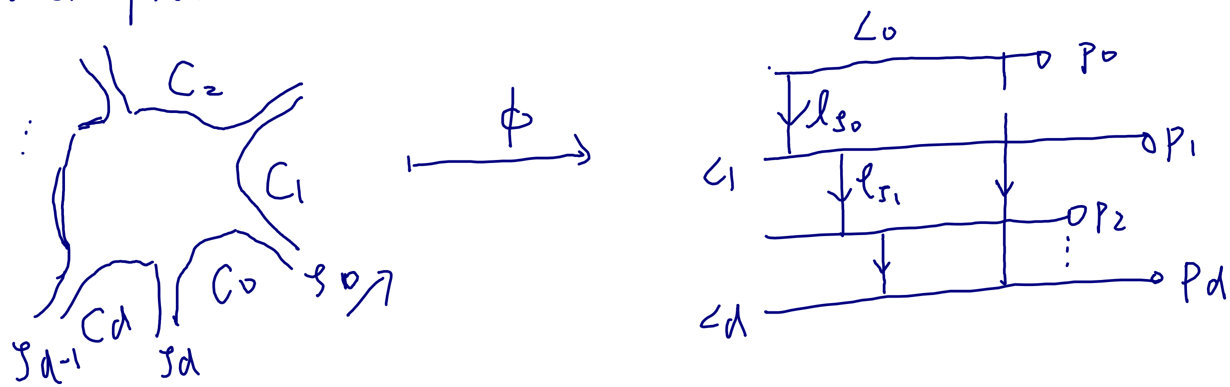
3° the deformation of R_α under MCE $\gamma_y \cong RS(JW)$

8) Fukaya Cat. of LG model

(X, h, W) tame system

Moduli problem:



$$\begin{cases} \bar\partial_J \phi + X_{K,\delta}(\phi) = 0 \\ \phi|_C \in L_C, \ \forall \text{ component } C \subset \partial\Sigma \\ \lim_{s \to \pm\infty} \phi(\varepsilon_\xi(s,\cdot)) = \ell_\xi, \ \forall \xi \in \Sigma \end{cases}$$

$\downarrow$ flow lines

Composition $\mu^d: \mathrm{Hom}(L_0^\#, L_1^\#) \otimes \cdots \otimes \mathrm{Hom}(L_{d-1}^\#, L_d) \longrightarrow \mathrm{Hom}(L_0^\#, L_d^\#)$

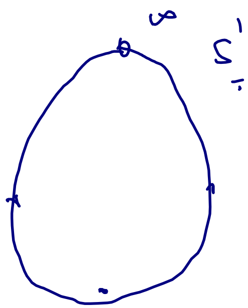
$$\mu^d(\ell^1, \ldots, \ell^d) = \sum_{\ell \in S_W} \pm \langle \ell, \ell^1 \cdots \ell^d \rangle \cdot \ell$$

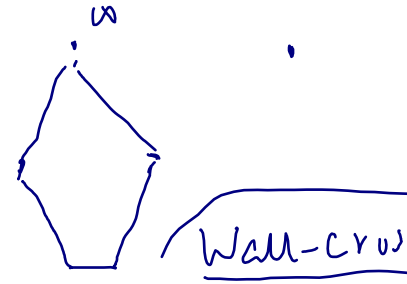
[FJY, 2018]

$$CF(W) = \bigoplus_{L_0^\#, L_1^\# \text{ LG branes}} CF(L_0^\#, L_1^\#)$$

Thm ($CF(W)$, μ_*) is a A_∞-Category.

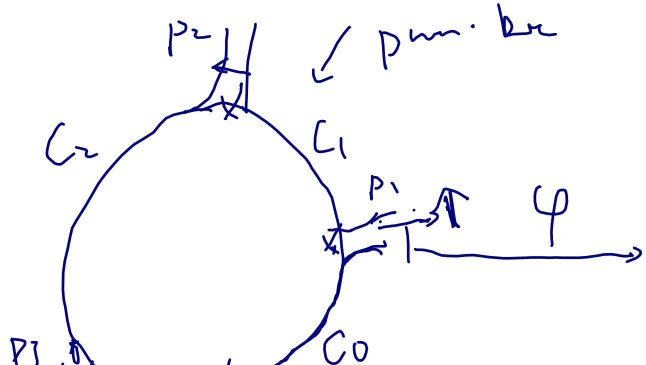
Thank you!





Wall-crossing

multi-punctured



$$\partial\varphi + h^{ij}\frac{\partial W}{\partial x^i} = 0$$

$$\Phi(C_i) \subset L_i$$

$$\Phi(p_i) \longmapsto l_i$$