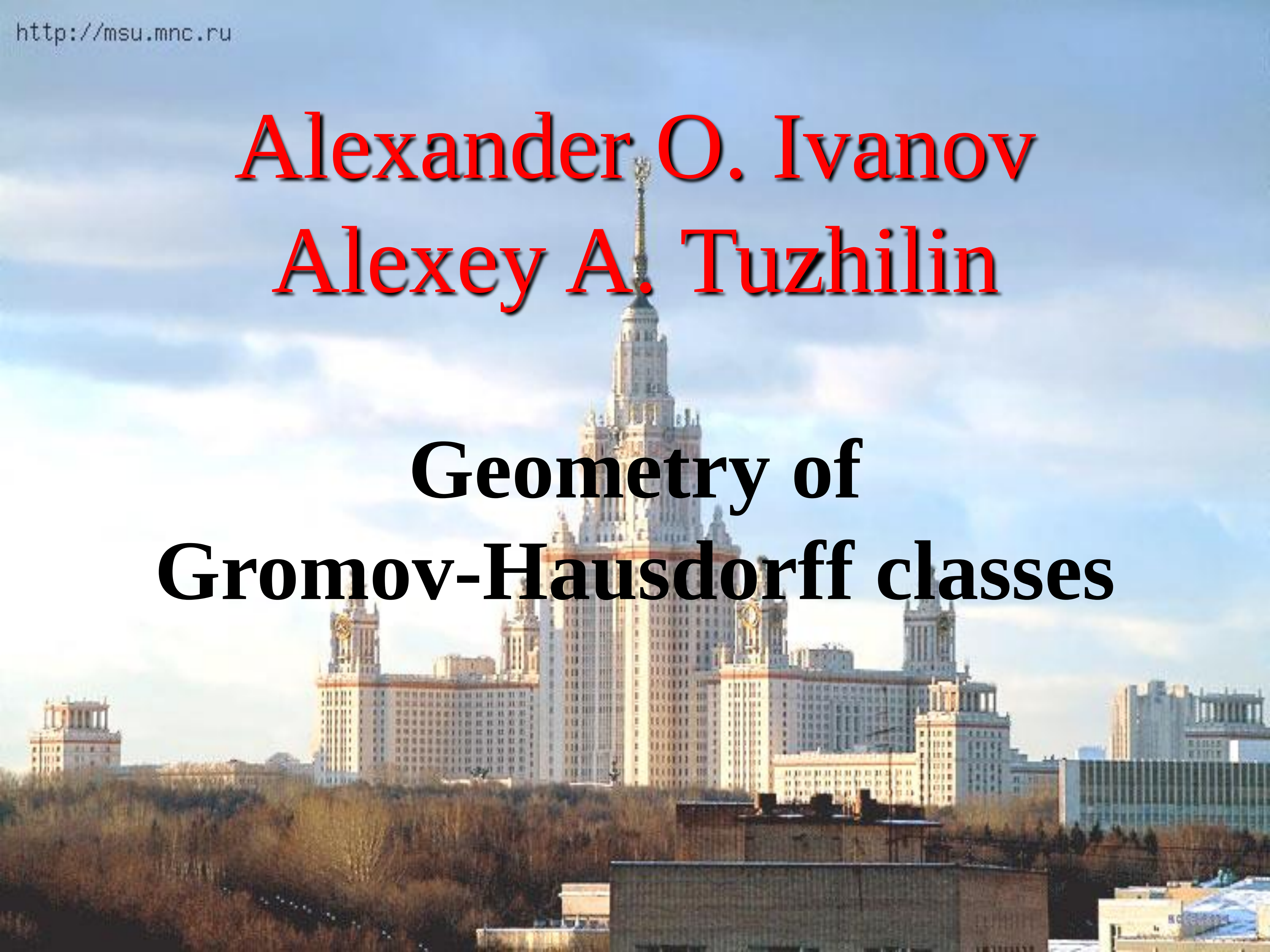

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Geometry of Gromov-Hausdorff classes

Preliminaries

Sets and Proper Classes (NBG).

Von Neumann–Bernays–Gödel set theory

All objects of NBG are called classes. There are two types of classes:
- a set $\mathcal{A}$: there exists a class C such that $\mathcal{A} \in C$;
- a proper class $\mathcal{A}$: for any class C it holds $\mathcal{A} \notin C$.

Remark. Any element of a class is a set;
$\mathcal{V} = \{\mathcal{A} : \mathcal{A} = \mathcal{A}\}$ is the class of all sets, $\mathcal{V}$ is a proper class.

For all classes $\mathcal{A}$, $\mathcal{B}$ it is defined $\mathcal{A} \times \mathcal{A}$, $f : \mathcal{A} \to \mathcal{B}$, etc., in particular, we can speak about distance function on a class:

$\rho : \mathcal{A} \times \mathcal{A} \to [0, \infty]$ is a distance if

$\rho(x, x) = 0$ and $\rho(x, y) = \rho(y, x)$ for all $x, y \in \mathcal{A}$.
If ρ satisfies the triangle inequality

$\rho(x, y) + \rho(y, z) \geq \rho(x, z)$ for all $x, y, z \in \mathcal{A}$,

then ρ is an extended pseudometric.
Pseudometric is an extended metric if $\rho(x, y) = 0$ is equivalent to $x = y$.
If $\rho(x, y) < \infty$, then we remove the word "extended", sometimes we also add the word "finite".

To simplify notations, we shall write $|x\,y|$ instead of $\rho(x, y)$.

However, we cannot define topology τ on a proper class $\mathcal{A}$ in the standard way: otherwise, $\mathcal{A} \in \tau$, hence $\mathcal{A}$ is a set.

To overcome, let $\mathcal{A}_n \subset \mathcal{A}$ consist of all elements (sets) whose cardinality is at most n. We get a "filtration" of $\mathcal{A}$ w.r.t. various n :

$$\ldots\ \mathcal{A}_m \subset \mathcal{A}_n\ \ldots,\ \ldots\ m < n\ \ldots$$

Suppose that each $\mathcal{A}_n$ is a set. We call such $\mathcal{A}$ set-filtered.

Let $\mathcal{A}$ be set-filtered.
We define a topology τ on $\mathcal{A}$ as a mapping $\tau : n \to \tau_n$, where τ_n is a topology on $\mathcal{A}_n$ such that τ_m is induced from τ_n for all $m < n$ (in this case we say that the topologies τ_n are consistent).
We call a set-filtered class $\mathcal{A}$ with a topology τ a topological class.

If ρ is a pseudometric on a set-filtered class $\mathcal{A}$, then ρ induces the corresponding topology τ_n on $\mathcal{A}_n$ for each n, and the topologies τ_n are obviously consistent. We call such τ a pseudometric topology.

Let X be a set, $\mathcal{A}$ a class, and $f: X \to \mathcal{A}$ a mapping.

Observation. The image $f(X)$ belongs to some $\mathcal{A}_n$.
Proof. By axiom of replacement, $f(X)$ is a set.
By axiom of union, $B := \bigcup f(X) = \bigcup \{a \in f(X)\}$ is a set.
Let n be the cardinality of B.
Since for each $x \in X$ it holds $f(x) \subset B$, then $f(x) \in \mathcal{A}_n$.

As a corollary, for each topological space X and any topological class $\mathcal{A}$, one can define the notion of continuous mapping:

$f: X \to \mathcal{A}$ is continuous if

the mapping $f: X \to \mathcal{A}_n$ is continuous for some (and, therefore, for each) $\mathcal{A}_n \supset f(X)$.

In particular, for topological class we can speak about continuous curves and linear connectivity. If a class $\mathcal{A}$ is endowed with an extebded pseudometric ρ, then we can measure the length $|\gamma|$ of any continuous curve γ and define the notion of intrinsic distance function: ρ is called intrinsic if for any $x, y \in \mathcal{A}$ it holds

$$|xy| = \inf\{|\gamma| : \gamma \text{ joins } x \text{ and } y\}.$$

Let $\mathcal{VGH}$ be the class of all metric spaces, and $\mathcal{GH}$ be the class of all isometry classes of metric spaces.

Since for each set one can define a metric (e.g., let us set all non-zero distances to 1), then the both $\mathcal{VGH}$ and $\mathcal{GH}$ are proper classes.

Observation. For any cardinal number, $\mathcal{GH}_n$ is a set.
Proof. The class of all cardinals not exceeding n is a set.
For each set X of cardinality n, the class of all metrics on X considered upto isometry is a subset of the set of all subsets of $X \times X \times [0, \infty)$. It remains to use axiom of union.

Thus, $\mathcal{GH}$ is set-filtered.

Now, we define a distance function on $\mathcal{GH}$.
To do that, we recall some necessary notions.

Hausdorff Distance.

Let X be a metric space, $x \in X$, $A \subset X$ is nonempty, $r > 0$, $s \geq 0$, then

$U_r(x) = \{z \in X : |xz| < r\}$ is open ball,
$B_s(x) = \{z \in X : |xz| \leq s\}$ is closed ball,
$|xA| = |Ax| = \inf\{|xa| : a \in A\}$,
$U_r(A) = \{x \in X : |xA| < r\}$ is open r-neighborhood of A,
$B_s(A) = \{x \in X : |xA| \leq s\}$ is closed s-neighborhood of A.

For nonempty $A, B \subset X$ let

$$\begin{aligned} d_H(A, B) &= \inf\{r : A \subset U_r(B) \text{ and } B \subset U_r(A)\} \\ &= \inf\{s : A \subset B_s(B) \text{ and } B \subset B_s(A)\} \\ &= \max\{\sup\{|aB| : a \in A\}, \sup\{|Ab| : b \in B\}\} \end{aligned}$$

be the Hausdorff distance between A and B.

Proposition. Let $\mathcal{H}(X)$ be the set of all nonempty closed bounded $A \subset X$. Then d_H is a metric on $\mathcal{H}(X)$. The space $\mathcal{H}(X)$ is complete (totally bounded, compact) iff X is.

Gromov-Hausdorff Distance.

Let X and Y be metric spaces, then

$$d_{GH}(X, Y) = \inf\{r : d_H(X', Y') \le r;\ X', Y' \subset Z;\ Z \in \mathcal{GH};\ X \approx X';\ Y \approx Y'\},$$

where for metric spaces U and V, $U \approx V$ means that U is isometric to V.

Evidently, $d_{GH}(X, X) = 0$ and $d_{GH}(X, Y) = d_{GH}(Y, X)$, hence

$$d_{GH} : \mathcal{VGH} \times \mathcal{VGH} \to [0, \infty]$$ is a distance function.

The value $d_{GH}(X, Y)$ is called the Gromov-Hausdorff distance between the metric spaces X and Y.

Theorem. The distance function d_{GH} satisfies the triangle inequality, thus, d_{GH} is an extended pseudometric on $\mathcal{VGH}$.

Theorem. If $X \approx Y$, then $d_{GH}(X, Y) = 0$, thus, d_{GH} generates an extended pseudometric on $\mathcal{GH}$ (we denote it by the same d_{GH}).

Remark. In what follows, we shell identify the elements of $\mathcal{GH}$ (the isometry classes) with their representatives (the metric spaces). Thus, we say that $\mathcal{GH}$ is the class of all metric spaces *considered upto isometry*.

There are a few important subclasses in $\mathcal{GH}$. One of them consists of all bounded metric spaces. We denote it by $\mathcal{B}$. We shall see that the restriction of d_{GH} to $\mathcal{B}$ is a finite pseudometric.

Another important class consists of all compact metric spaces. We denote it by $\mathcal{M}$. Indeed, $\mathcal{M}$ is a set of cardinality continuum.

Theorem. Let $\mathcal{M}$ be the set of isometry classes of compact metric spaces. Then

1) d_{GH} is a metric on $\mathcal{M}$;
2) $\mathcal{M}$ is contractible, path-connected, complete, separable, not locally compact.

$\mathcal{M}$ with d_{GH} is called the Gromov-Hausdorff space.

Correspondences and Gromov-Hausdorff Distance.

Let X, Y be sets,
each $\sigma \subset X \times Y$ is called a relation between X and Y,
$\mathcal{P}_0(X, Y)$ is the set of all nonempty relations between X and Y.

$\pi_X : X \times Y \to X$, $\pi_X(x, y) = x$; $\pi_Y : X \times Y \to Y$, $\pi_Y(x, y) = y$,
$R \subset X \times Y$ is a correspondence iff $\pi_X|_R$ and $\pi_Y|_R$ are surjections,
$\mathcal{R}(X, Y)$ is the set of all correspondences between X and Y.

X, Y are metric spaces, $\sigma \in \mathcal{P}_0(X, Y)$, then the distortion of σ is

$$\operatorname{dis}\sigma = \sup\big\{\big||xx'| - |yy'|\big| : (x, y), (x', y') \in \sigma\big\}.$$

Theorem. For any metric spaces X and Y we have

$$2\, d_{GH}(X,Y) = \inf\big\{\operatorname{dis} R : R \in \mathcal{R}(X, Y)\big\}.$$

Optimal and ε-Optimal Correspondences.

For any metric spaces X and Y with $d_{GH}(X,Y) < \infty$, and any $\varepsilon > 0$, a correspondence $R \in \mathcal{R}(X, Y)$ is called ε-optimal, if

$$\operatorname{dis} R - 2\, d_{GH}(X,Y) < 2\varepsilon.$$

If $\operatorname{dis} R = 2d_{GH}(X,Y)$, then the correspondence R is called optimal.

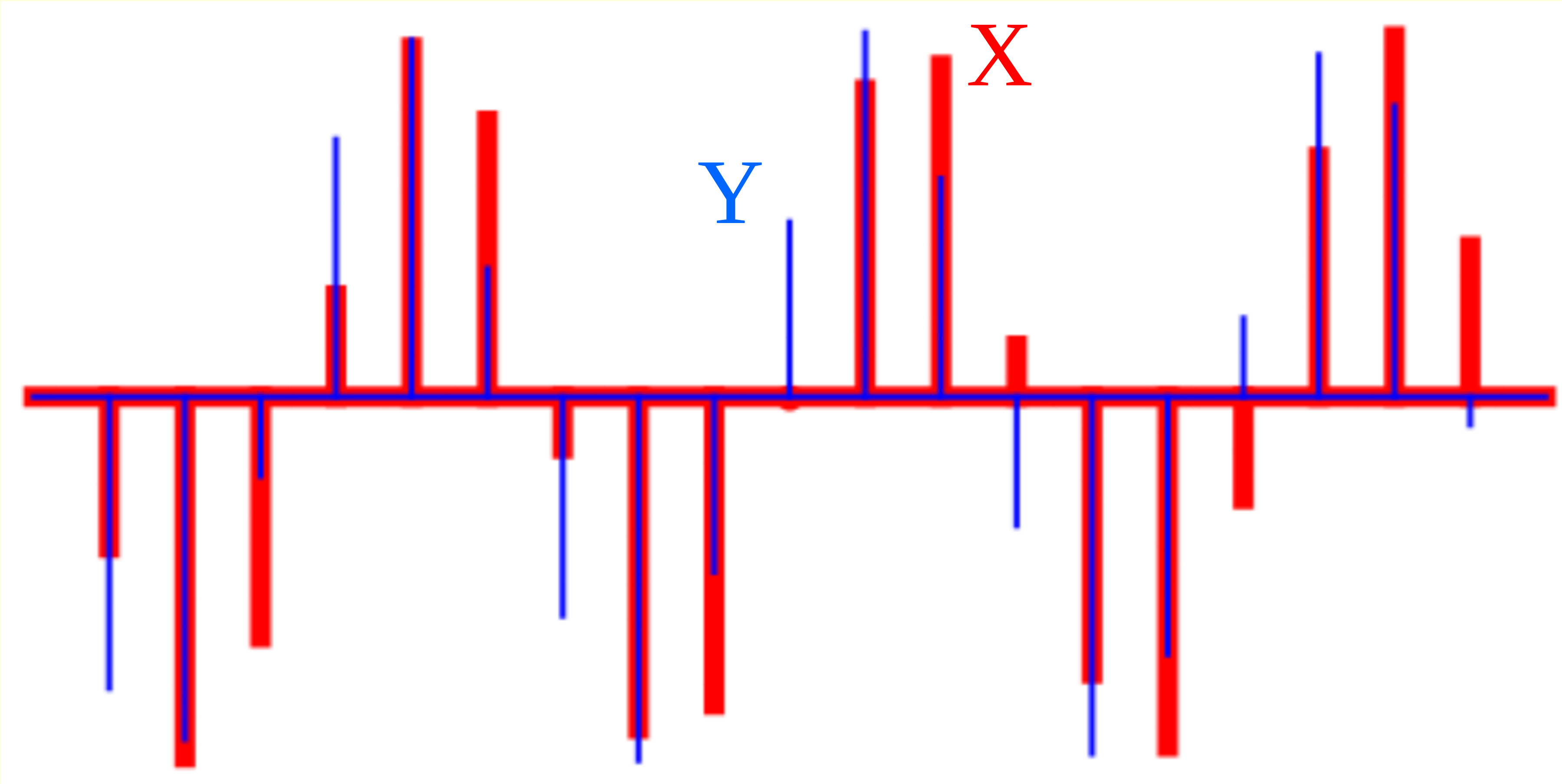
We put

$$\mathcal{R}_{\varepsilon\text{-opt}}(X, Y) = \{R \in \mathcal{R}(X, Y) : R \text{ is } \varepsilon\text{-optimal}\},$$

$$\mathcal{R}_{\text{opt}}(X, Y) = \{R \in \mathcal{R}(X, Y) : R \text{ is optimal}\}.$$

Clearly that $\mathcal{R}_{\varepsilon\text{-opt}}(X, Y) \neq \emptyset$, however $\mathcal{R}_{\text{opt}}(X, Y)$ can be empty.

Example. Let X and Y be subsets of the Euclidean plane with coordinates (x, y). The both consist of the abscissa $y = 0$ and some vertical segments between integer points $(n, 0)$ and points $(n, f(n))$ for some $f(n)$. Namely, for X we take $f(n) = \sin(n)$, and for Y we put $f(n) = \sin(n+1/2)$.



One can show that X and Y are not isometric, however, for any $\varepsilon > 0$ there exists an integer shift $\tau = (m,0)$ such that $d_H(\tau X, Y) < \varepsilon$, thus $d_{GH}(\tau X, Y) = 0$. If there exists an optimal correspondence $R \in \mathcal{R}(X, Y)$, then $\operatorname{dis} R = 0$, hence R is an isometry, a contradiction.

Shortest and ε-Shortest Geodesics.

Let $\mathcal{A}$ be a set-filtered class endowed with an extended pseudometric.

Given $\varepsilon > 0$, a curve γ in $\mathcal{A}$ joining X and Y from $\mathcal{A}$ is called ε-shortest geodesic, if $|\gamma| - |XY| < \varepsilon$.

If $|\gamma| = |XY|$, then γ is called shortest geodesic.

The extended pseudometric on $\mathcal{A}$ is called intrinsic, if for any $X, Y \in \mathcal{A}$ with $|XY| < \infty$, and $\varepsilon > 0$, there exists an ε-shortest geodesic in $\mathcal{A}$ joining X and Y.

If for any $X, Y \in \mathcal{A}$ with $|XY| < \infty$ there exists a shortest geodesic in $\mathcal{A}$ joining X and Y, then the extended pseudometric on $\mathcal{A}$ is called strictly intrinsic, and $\mathcal{A}$ itself is called a geodesic class.

Now, let us take as $\mathcal{A}$ the classes $\mathcal{GH}$, $\mathcal{B}$, and $\mathcal{M}$.

Gromov-Hausdorff distance on $\mathcal{GH}$ and $\mathcal{B}$ is Intrinsic.

Theorem (A.Ivanov, AT). Let $X, Y \in \mathcal{GH}$ and $d_{GH}(X,Y) < \infty$. Consider an arbitrary correspondence $R \in \mathcal{R}(X, Y)$ with $\operatorname{dis} R < \infty$, and for any $(x,y), (x',y') \in R$ and $t \in (0,1)$ let us put

$$|(x, y)(x', y')|_t = (1- t)\,|xx'| + t\,|yy'|.$$

Then $|\,.\,.\,|_t$ is a metric on R.

Let R_t be R with the metric $|\,.\,.\,|_t$, $t \in (0,1)$, and $R_0 = X$, $R_1 = Y$, then $\gamma(t) = R_t$ is an continuous curve in $\mathcal{GH}$. If R is ε-optimal, then γ is an ε-shortest geodesic. Thus, the Gromov-Hausdorff distance is intrinsic on $\mathcal{GH}$.

Moreover, if $X, Y \in \mathcal{B}$, then the curve γ belongs to $\mathcal{B}$, therefore, the Gromov-Hausdorff distance is intrinsic on $\mathcal{B}$ as well.

Problem. Since the Gromov-Hausdorff distance satisfies the triangle inequality, then it is the same between any representatives of isometry classes and, hence, it generates an extended metric on $\mathcal{GH}$ factorized by zero-distance equivalence. Is it true that the obtained class is geodesic?

The Gromov-Hausdorff Space $\mathcal{M}$ is Geodesic.

For any $X, Y \in \mathcal{GH}$, the space $X \times Y$ has natural topology, thus for any relation $\sigma \subset X \times Y$ its closure σ^c is defined. It is easy to see that for any $R \in \mathcal{R}(X, Y)$ it holds $\operatorname{dis} R = \operatorname{dis} R^c$. Denote by $\mathcal{R}^c(X, Y)$ the subset of $\mathcal{R}(X, Y)$ consisting of all closed correspondences. We immediately get

Theorem. For any metric spaces X and Y we have

$$2\, d_{GH}(X,Y) = \inf\{\operatorname{dis} R : R \in \mathcal{R}^c(X, Y)\}.$$

If X and Y are compact, i.e., $X, Y \in \mathcal{M}$, then all R_t defined above are also compact, so the curve $\gamma(t) = R_t$ belongs to $\mathcal{M}$. Moreover, $\mathcal{R}^c(X, Y)$ is compact, and $\operatorname{dis} : \mathcal{R}^c(X, Y) \to [0, \infty)$ is continuous, thus $\mathcal{R}^c(X, Y) \cap \mathcal{R}_{opt}(X, Y) \neq \emptyset$.

Theorem (S.Chowdhury, F.Mémoli; A.Ivanov, AT).
For any $X, Y \in \mathcal{M}$ there exists an optimal $R \in \mathcal{R}^c(X, Y)$. For such R, the curve $\gamma(t) = R_t$ is a shortest geodesic in $\mathcal{M}$ joining X and Y, and $|\gamma| = d_{GH}(X,Y)$. Therefore, the metric space $\mathcal{M}$ is geodesic.

Each shortest geodesic $\gamma(t) = R_t$ will be called an R-geodesic.

Some Geometry of $\mathcal{GH}$.

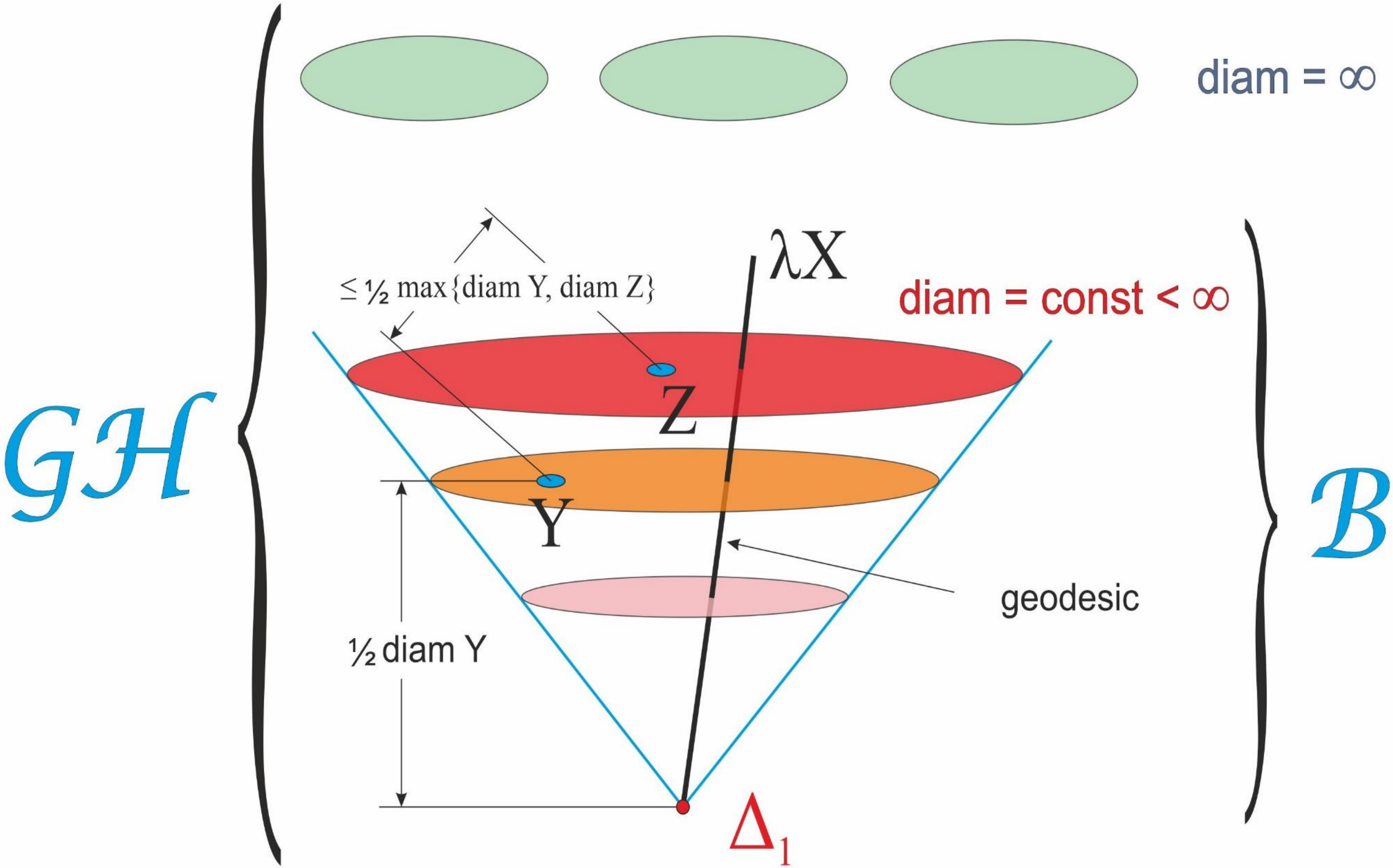
For a metric space X and $\lambda > 0$, we denote by λX the metric space with the same set X and with the distance $\lambda|xy|$ for each $x, y \in X$. For $\lambda = 0$ we define λX to be the one-point space Δ_1.
More generally, let Δ_n be n-point space with all nonzero distances equal 1.

For any metric space X we define its diameter as

$$\operatorname{diam} X = \sup\{|xy| : x, y \in X\}.$$

Theorem. For any $X, Y \in \mathcal{GH}$ we have

1) $2d_{GH}(\Delta_1, X) = \operatorname{diam} X$;
2) if $\min\{\operatorname{diam} X , \operatorname{diam} Y\} < \infty$, then
$$2d_{GH}(X, Y) \geq |\operatorname{diam} X - \operatorname{diam} Y|$$ (the triangle inequality);
3) $2d_{GH}(X, Y) \leq \max\{\operatorname{diam} X, \ \operatorname{diam} Y\}$ (ultrametric-like inequality);
4) if $\operatorname{diam} X < \infty$, then for any $\lambda, \mu > 0$ we have
$$2d_{GH}(\lambda X, \mu X) = |\lambda - \mu| \operatorname{diam} X,$$ i.e., $t \to t X$ is a geodesic;
5) for any $\lambda > 0$ we have $d_{GH}(\lambda X, \lambda Y) = \lambda\, d_{GH}(X, Y)$,
   i.e., $X \to \lambda X$ is a homothety centered at Δ_1. Thus, $\mathcal{B}$ is a cone with the origin at Δ_1, foliated by spheres $\operatorname{diam} = \text{const}$ centered at Δ_1.



diam = ∞
λX
≤ ½ max{diam Y, diam Z}
diam = const < ∞
Z
Y
GH
B
geodesic
½ diam Y
Δ1

Geometry of Metric Segments in $\mathcal{GH}$.

Let $\mathcal{A}$ be a class endowed with an extended pseudometric, and $X, Y \in \mathcal{A}$.
We say that $Z \in \mathcal{A}$ is located between X and Y if $|XZ| + |ZY| = |XY|$.
The class $[X, Y]$ consisting of all Z located between X and Y we call a metric segment. A metric segment $[X, Y]$ is called nondegenerated if $|XY| > 0$.

Now, let $X, Y \in \mathcal{GH}$. Is $[X, Y]$ a set or a proper class?

Theorem (O.Borisova). For any $X, Y \in \mathcal{B}$, the metric segment $[X, Y]$ is a proper class. Moreover, if $X, Y \in \mathcal{M}$, then the restriction of the metric segment $[X, Y]$ to $\mathcal{M}$, i.e., $[X, Y] \cap \mathcal{M}$, is not compact.

Remark. It is very likely that the same is true for any $X, Y \in \mathcal{GH}$.
Let $X, Y \in \mathcal{M}$, then the union of all R–geodesics joining X and Y we denote by $[X, Y]_c$. Note that $[X, Y]_c \subset \mathcal{M}$.

Theorem (D.Klibus). For any $X, Y \in \mathcal{M}$, the set $[X, Y]_c$ is compact.

The set $[X, Y]_c \subset \mathcal{M}$ we call a compact metric segment.

Extention of Metric Segments in $\mathcal{GH}$.

Let $\mathcal{A}$ be a class endowed with an extended pseudometric, and $X, Y \in \mathcal{A}$.
We say that a metric segment $[X, Y] \subset \mathcal{A}$ can be extended beyond Y if there exists $Z \in \mathcal{A}$ such that $|ZY| > 0$ and Y is located between X and Z.
Clearly that if $|XY| = 0$, then each Z extends $[X, Y]$ beyond the both X and Y.

Now, let $\mathcal{A} = \mathcal{B}$. Recall that for any $X, Y \in \mathcal{B}$ it holds

1) $2d_{GH}(\Delta_1, X) = \operatorname{diam} X$;
2) $2d_{GH}(X, Y) \ge |\operatorname{diam} X - \operatorname{diam} Y|$;
3) $2d_{GH}(X, Y) \le \max\{\operatorname{diam} X, \operatorname{diam} Y\}$.

If $\operatorname{diam} Y$ is fixed, and $\operatorname{diam} X \le \operatorname{diam} Y$, then the maximal value of $2d_{GH}(X, Y)$ is $\operatorname{diam} Y$.
We say that Y is hyperextreme w.r.t X, if

$$d_{GH}(X, Y) > 0 \text{ and } 2d_{GH}(X, Y) = \operatorname{diam} Y \ge \operatorname{diam} X.$$

If $\operatorname{diam} X$ and $\operatorname{diam} Y$ are fixed, and $\operatorname{diam} X \le \operatorname{diam} Y$, then the minimal value of $2d_{GH}(X, Y)$ is $\operatorname{diam} Y - \operatorname{diam} X$.
We say that Y is subextreme w.r.t. X, if

$$d_{GH}(X, Y) > 0 \text{ and } 2d_{GH}(X, Y) = \operatorname{diam} Y - \operatorname{diam} X.$$

A metric segment $[X, Y] \subset \mathcal{B}$ is called extreme, if X and Y are mutually hyperextreme ($2d_{GH}(X, Y) = \operatorname{diam} Y = \operatorname{diam} X$).

We also need a few more characteristics of metric spaces.

Let X be an arbitrary metric space, m be a cardinal number, $\mathcal{C}_m(X)$ the set of m-elements coverings of X, and $\mathcal{D}_m(X)$ the set of m-elements partitions of X.

For $C = \{C_i\}_{i\in I} \in \mathcal{C}_m(X)$ and $D = \{D_i\}_{i\in I} \in \mathcal{D}_m(X)$, we put

$$\operatorname{diam} C = \sup_{i\in I} \operatorname{diam} C_i \quad \text{and} \quad \alpha(D) = \inf\big\{|D_i D_j| : i \neq j\big\}.$$

Theorem (S.Borzov, A.Ivanov, AT). Let $X, Y \in \mathcal{B}$ such that the metric segment $[X, Y]$ is extreme. Suppose also that the following conditions hold:

(1) there exists a partition $D_X \in \mathcal{D}_n(X)$ such that $\alpha(D_X) > 0$;

(2) there exists a covering $C_Y \in \mathcal{C}_m(Y)$ such that $\operatorname{diam} C_Y < \operatorname{diam} Y$;

(3) $m \le n$.

Then the metric segment $[X, Y]$ cannot be extended beyond Y.

In particular, if $X, Y \in \mathcal{M}$, then no one shortest geodesic joining X and Y can be extended beyond Y up to a shortest geodesic.

Examples.

1) Let $X, Y \in \mathcal{B}$ and Y is subextreme w.r.t. X, then $[X, Y]$ can be extended beyond Y. If $X \neq \Delta_1$, then the $[X, Y]$ can be also extended beyond X.

2) For any $X \in \mathcal{GH}$ and any cardinal m we put $\alpha_m(X) = \sup\{\alpha(D) : D \in \mathcal{D}_m(X)\}$. Thus, for any $X \in \mathcal{B}$, any cardinal $m \le \#X$, and any $\lambda \ge \operatorname{diam} X + \alpha_m(X)$, $[X, \lambda\Delta_m]$ can be extended beyond $\lambda\Delta_m$ upto $\lambda'\Delta_m$ for each $\lambda' > \lambda$.

3) Let $X = \lambda\Delta_n$, $Y = \lambda\Delta_m$, and $m < n$, then $[X, Y]$ cannot be extended beyond its ends.

4) Let $X = \Delta_2$, $Y = [0, 1]$, then $[X, Y]$ can be extended beyond X, but cannot beyond Y.

5) Let $X = \Delta_k$, $1 < k < \infty$, $Y \subset \mathbb{R}^{k-1}$ – a convex body with smooth boundary and $\operatorname{diam} Y = 1$. Taking into account Hugo Hadwiger result on Borsuk problem (that Y can be covered by k subsets of smaller diameter), we get that $[X, Y]$ cannot be extended beyond Y.

6) There is no a metric segment $[X, Y] \subset \mathcal{B}$ that can be extended upto infinity beyond the both of its ends.

Isometries of $\mathcal{GH}$.

Since we consider isometry classes of metric spaces, it seems that the space $\mathcal{GH}$ is very heterogeneous.

Conjecture (Stavros Iliadis). There is no nontrivial self-isometries of $\mathcal{M}$.

The question was discussed in https://mathoverflow.net in 2015. The question about nonexistence of nontrivial self-isometries was posed by Noah Schweber, and it was suggested the positive answer and a sketch of the proof by George Lowther. However, the proof is very draft.

Theorem (G.Lowther, A.Ivanov, AT). $\mathrm{Iso}(\mathcal{M})$ is trivial.

One can show that each isometry f of $\mathcal{GH}$ preserves the one-point metric space Δ_1. This implies that f preserves $\mathcal{M}$, $\mathcal{B}$, and the class of all unbounded metric spaces.

Problem. Is it true that $\mathrm{Iso}(\mathcal{GH})$ is trivial?

Huge Number of Local Isometries.

Let $M = \{1,\dots,n\} \in \mathcal{M}$ be a finite metric space. We say that M is in general position if all triangle inequalities are strict and $\mathrm{Iso}(M)$ is trivial.

Proposition (A.Filin, A.Ivanov, AT). For sufficiently small $\varepsilon > 0$ and any $X \in U_\varepsilon(M) \subset \mathcal{M}$ there exists a partition $\{X_i\}_{i=1,\dots,n}$ of X such that $\mathrm{diam}\, X_i < \varepsilon$ for each $i=1, \dots, n$, and for any $x_i \in X_i$ and $x_j \in X_j$ it holds

$$\big||x_i x_j| - |ij|\big| < \varepsilon.$$

For M in general position, the partition $\{X_i\}_{i=1,\dots,n}$ is uniquely defined.

Definition. The partition $\{X_i\}_{i=1,\dots,n}$ from Proposition is called canonical.

Construction. Let $M = \{1,\dots,n\}$ and $N = \{1,\dots,n\}$ be general position metric spaces with distance functions $|\cdot|_M$ and $|\cdot|_N$, resp. For sufficiently small $\varepsilon > 0$, any $X \in U_\varepsilon(M) \subset \mathcal{M}$, and the canonical partition $\{X_i\}_{i=1,\dots,n}$ of X we define a new distance function ρ on the set X as follows:

$$\rho(x_i, x_j) = |x_i x_j| - |\,i\,j\,|_M + |\,i\,j\,|_N.$$

Theorem (A.Ivanov, AT). The distance function ρ is a metric on X, and if Y is the set X with the metric ρ, then $Y \in U_\varepsilon(N) \subset \mathcal{M}$. Moreover, the mapping f taking each X to such Y, generates an isometry

$$f : U_\varepsilon(M) \to U_\varepsilon(N).$$

Thus, sufficiently small neighborhoods of any two generic position n-points metric spaces are isometric.

Now, if $|i\,j|_N = |\sigma(i)\sigma(j)|_M$ for some permutation $\sigma \in S_n$, then we obtain an isometry $f_\sigma: U_\varepsilon(M) \to U_\varepsilon(M)$ as above.

Proposition (A.Ivanov, AT). Let $M = \{1,\ldots,n\}$, $n \geq 3$, be a general position metric space, and $\sigma, \tau \in S_n$ be permutations. Then the isometries f_σ and f_τ are distinct. Thus, the isometry group of the neighborhood $U_\varepsilon(M)$ contains a subgroup isomorphic to S_n.

Problem. What are other local isometries of $\mathcal{GH}$?

Finite-spaces universality of $\mathcal{GH}$.

We put $\mathcal{M}_n = \{X \in \mathcal{M} : \#X \le n\}$ and $\mathcal{M}_{[n]} = \{X \in \mathcal{M} : \#X = n\} \subset \mathcal{M}_n$.
Denote by $\mathbb{R}^n_\infty$ the space $\mathbb{R}^n$ with the metric generated by the max-norm.

Theorem (A.Ivanov, AT). Let $M \in \mathcal{M}_{[n]}$ be in general position. Then there exists a neighborhood $U \subset \mathcal{M}_{[n]}$ of M such that U is isometric to an open subset of $\mathbb{R}^n_\infty$. The size of U can be estimated in terms of the geometry of M.

Let $X = \{x_1, \dots x_n\}$ be a finite metric space.
Recall that the Kuratowski mapping $\nu : X \to \mathbb{R}^n_\infty$ is

$$\nu : x_i \to (|x_i x_1|, \dots , |x_i x_n|).$$

It is well-known that ν is an isometric embedding.
Taking sufficiently large U from Theorem, we get

Corollary (A.Ivanov, S.Iliadis, AT).
Each finite metric space X can be isometrically embedded into $\mathcal{GH}$.

Problem. Describe all metric spaces which can be isometrically embedded into $\mathcal{GH}$.

Path Connectivity of Spheres in $\mathcal{GH}$.

Below we present some results concerning path-connectivity of spheres in $\mathcal{M}$. The general question is open.

Theorem (R.Tsvetnikov). Any sphere in $\mathcal{M}$ with center at the one-point metric space Δ_1 is path-connected.

Theorem (R.Tsvetnikov). For any $X \in \mathcal{M}$ and any $r > \operatorname{diam} X$ the sphere with center at X and radius r is path-connected.

Theorem (R.Tsvetnikov). For any finite metric space $X \in \mathcal{M}$ there exists $\varepsilon > 0$ such that for any $0 \le r < \varepsilon$ the sphere with center at X and radius r is path-connected.

Problem. Complete investigation of path connectivity of spheres in $\mathcal{M}$. Investigate path-connectivity of spheres in $\mathcal{B}$ and $\mathcal{GH}$.

The Mapping $\mathcal{H}$.

Recall that for any non-empty metric space X, by $\mathcal{H}(X)$ we mean the set of all nonempty closed bounded $A \subset X$, and we endow $\mathcal{H}(X)$ with the Hausdorff distance. As we mentioned above, such $\mathcal{H}(X)$ is a metric space. Thus we obtained a mapping

$$\mathcal{H} : \mathcal{GH} \to \mathcal{GH}.$$

Notice that for a compact X, the space $\mathcal{H}(X)$ is compact as well, thus $\mathcal{M}$ is invariant under the mapping $\mathcal{H}$.

Theorem (I.Mikhailov). The restriction of $\mathcal{H}$ to $\mathcal{M}$ is 1-Lipschitz, i.e.,

$$d_{GH}(\mathcal{H}(X), \mathcal{H}(Y)) \le d_{GH}(X, Y).$$

Mikhailov presented many various pairs of compact metric spaces such that $\mathcal{H}$ preserves the distance for these pairs. The most obvious example is a pair of simplexes.

Problem. Is it true that $\mathcal{H}$ is isometric?

Gromov-Hausdorff Distances to Simplexes.

Let X be an arbitrary metric space and m a cardinal number. Recall that by $\mathcal{D}_m(X)$ we mean the set of m-elements partitions of X. Also, we introduced for each $D = \{X_i\}_{i\in I} \in \mathcal{D}_m(X)$

$$\operatorname{diam} D = \sup_{i\in I} \operatorname{diam} X_i\,, \quad \alpha(D) = \inf\{|X_i X_j| : i \neq j\}, \text{ and}$$

$$\alpha_m(X) = \sup\{\alpha(D) : D \in \mathcal{D}_m(X)\}.$$

Theorem (D.Grigor'ev, A.Ivanov, AT). For an arbitrary $X \in \mathcal{GH}$, a cardinal number $0 < m \leq \#X$, and a positive real number λ, we have

$$2\, d_{GH}(\lambda\Delta_m\,, X) = \inf\{\max(\operatorname{diam} D, \lambda - \alpha(D), \operatorname{diam} X - \lambda) : D \in \mathcal{D}_m(X)\}.$$

Corollary. For an arbitrary $X \in \mathcal{B}$ and a cardinal number $0 < m \leq \#X$,

if $\lambda \geq 2 \operatorname{diam} X$, then

$$2\, d_{GH}(\lambda\Delta_m\,, X) = \inf\{\lambda - \alpha(D) : D \in \mathcal{D}_m(X)\} = \lambda - \alpha_m(X);$$

if $0 < \lambda < \operatorname{diam} X$, then

$$2\, d_{GH}(\lambda\Delta_m\,, X) = \operatorname{diam} X \text{ iff } \operatorname{diam} D = \operatorname{diam} X \text{ for all } D \in \mathcal{D}_m(X).$$

Minimum Spanning Trees.

Let X be a finite metric space, then

$$\mathrm{mst}(X) = \inf\{|G| : G = (X, E) \text{ is a tree}\};$$

$G = (X, E)$ is a tree, $|G| = \mathrm{mst}(X)$, then G is called a

minimum spanning tree on X;

$$\mathrm{MST}(X) = \{G : G \text{ is a minimum spanning tree on } X\};$$

Evidently, $\mathrm{MST}(X) \neq \varnothing$ and $\#\mathrm{MST}(X)$ may be more than 1.

Let $\#X = n$. For each $G \in \mathrm{MST}(X)$ we define $s(G) = (s_1, \ldots, s_{n-1})$ to be the vector obtained from the lengths of edges of G by ordering them descending.

Proposition. For any finite metric space X and $G_1, G_2 \in \mathrm{MST}(X)$ it holds $s(G_1) = s(G_2)$.

The vector from the above proposition is called the mst-spectrum of X and is denoted by $s(X)$.

Recall that for partition $D = \{X_i\}_{i\in I} \in \mathcal{D}_m(X)$ we put

$$\alpha(D) = \inf\{|X_i X_j| : i \neq j\}, \quad \alpha_m(X) = \sup\{\alpha(D) : D \in \mathcal{D}_m(X)\}.$$

Theorem (A.Ivanov, AT (variational approach)).
If X is a metric space, $\#X = n$, $\sigma(X) = (\sigma_1, \dots, \sigma_{n-1})$, then $\sigma_{m-1} = \alpha_m(X)$.

Corollary (AT). For a finite metric space X, $\#X = n$,
$\sigma(X) = (\sigma_1, \dots, \sigma_{n-1})$, a positive integer $2 \le m \le n$,
and $\lambda \ge 2\,\mathrm{diam}\,X$, it holds

$$\sigma_{m-1} = \lambda - 2\, d_{GH}(\lambda\Delta_m, X).$$

Thus, we expressed the lengths of minimum spanning tree edges in terms of Gromov-Hausdorff distances from the vertex set of the tree to simplexes.

Generalized Borsuk Problem.

Let X be a bounded metric space of diameter d, and $m \leq \#X$ a cardinal number. We say that X can be partitioned into m subsets X_i of strictly smaller diameter if for some $\varepsilon > 0$ we have $\operatorname{diam} X_i \leq d - \varepsilon$ for all i.

The Generalized Borsuk Problem: Describe those metric spaces $X \in \mathcal{B}$ which can be partitioned into m subsets of strictly smaller diameter.

Some history:
1932, **Karol Borsuk**: the standard n-dimensional ball in Euclidean space $\mathbb{R}^n$ can be partitioned into $n + 1$ subsets, each of which has a smaller diameter than the ball.
Borsuk question: is the same true for any bounded subset of $\mathbb{R}^n$?
True: $n = 2$ (**Borsuk**), $n = 3$ (**Perkal**, **Eggleston**, **Grünbaum**, **Heppes**) smooth convex bodies (**Hadwiger**), centrally-symmetric bodies (**Riesling**), bodies of revolution (**Dekster**)
1993, **Kahn** and **Kalai**: in general, FALSE (for $n = 1325$ and $n > 2014$)
2013, **Bondarenko**: false for $n \geq 65$, **Jenrich** – for $n = 64$ (the best known)

Generalized Borsuk Problem.

Theorem (A.Ivanov, AT). For an arbitrary bounded metric space X, any cardinal number $m \le \#X$, and any $0 < \lambda < \mathrm{diam}\, X$, we have $2\, d_{GH}(\lambda\Delta_m , X) \le \mathrm{diam}\, X$. The space X

- can be partitioned into subsets of strictly smaller diameters if

$$2\, d_{GH}(\lambda\Delta_m , X) < \mathrm{diam}\, X, \text{ and}$$

- cannot be partitioned into subsets of strictly smaller diameters if

$$2\, d_{GH}(\lambda\Delta_m , X) = \mathrm{diam}\, X.$$

Recall that $\mathcal{M}_{[n]}$ is the set of all isometry classes of bounded metric spaces of cardinality n, endowed with Gromov-Hausdorff distance.

For $X \in \mathcal{M}_{[n]}$ and $r > 0$, we define r–sphere at X as

$$S_r\,(X) = \{Y \in \mathcal{M}_{[n]} : d_{GH}(X , Y) = r\}.$$

Corollary (A.Ivanov, AT). Fix an arbitrary $0 < \lambda < \mathrm{diam}\, X$. Then the set of all metric spaces of cardinality n and diameter $d > 0$, which cannot be partitioned into $m \le n$ subsets of strictly smaller diameter, equals

$$S_{d/2}\,(\Delta_1) \cap S_{d/2}\,(\lambda\Delta_m).$$

Clique Cover Number.

A subgraph of an arbitrary simple graph G is called clique, if any its two vertices are connected by an edge (the subgraph is a complete graph itself).

Remark. Each single-vertex subgraph is also a clique.

The family of vertex sets of all cliques in a graph G forms a cover of the graph G vertex set. The least possible number of cliques forming such cover is called the clique cover number of G and is often denoted $\theta(G)$. Finding a minimum clique cover is NP-hard problem.

Examples.
1) Empty graph (without edges) with n vertices: $\theta = n$;
2) Complete graph K_n with n vertices: $\theta = 1$;
3) Complete k-partite graph $K(n_1,\ldots,n_k)$, $k > 1$, with parts of sizes $n_1,\ldots,n_k$:

$$\theta = \max\{n_1,\ldots,n_k\};$$

4) Cycle graph C_n with n vertices:

$$\theta = n/2 \text{ for even } n, \text{ and } \theta = (n+1)/2 \text{ for odd } n.$$

Clique Cover Number.

Let $G = (V, E)$ be a finite graph. Fix two real numbers $0 < a < b \leq 2a$.
Define a metric on V as follows: the distance between adjacent vertices of G equals a, and non-adjacent vertices of G equals b.

Observation. The space V can be partitioned into m subsets of strictly less diameters iff $\theta(G) \leq m \leq \#V$, and cannot iff $m < \theta(G)$.

Corollary (A.Ivanov, AT). Let m be the greatest positive integer such that $2\, d_{GH}(a\Delta_m , V) = b$, then $\theta(G) = m + 1$
(if there is no such m, then we put $m = 0$).

Example. 1) For empty n-graph, $V = b\Delta_n$, and $2\, d_{GH}(a\Delta_m , b\Delta_n) < b$ starting from $m = n$, thus $\theta = n$.
2) Let $G = K_n$, then $V = a\Delta_n$ and $2\, d_{GH}(a\Delta_m , a\Delta_n) < b$ for all positive m, then $m = 0$ and $\theta = m + 1 = 1$.
(3 and 4) For the corresponding metric spaces V we obtained exact values for $d_{GH}(a\Delta_m , V)$.

Chromatic Number.

A chromatic number of a simple graph G is the smallest numbers of colors to get an admissible coloring: adjacent vertices have different colors. The chromatic number of G is sometimes denoted by $\gamma(G)$.

The dual graph G' to G is the graph with the same vertex set, and with edges which join all non-adjacent vertices in G, and only them. It is well-known that $\gamma(G) = \theta(G')$. Thus, we have

Examples.

1) Empty graph (without edges) with n vertices: $\gamma = 1$;
2) Complete graph K_n with n vertices: $\gamma = n$;
3) Star graph with $n > 1$ vertices: $\gamma = 2$;
4) Wheel graph with $n > 1$ vertices:

   $\gamma = 3$ for even n, and $\gamma = 2$ for odd n;
5) Cycle graph C_n with $n > 1$ vertices:

   $\gamma = 2$ for even n, and $\gamma = 3$ for odd n.

Chromatic Number.

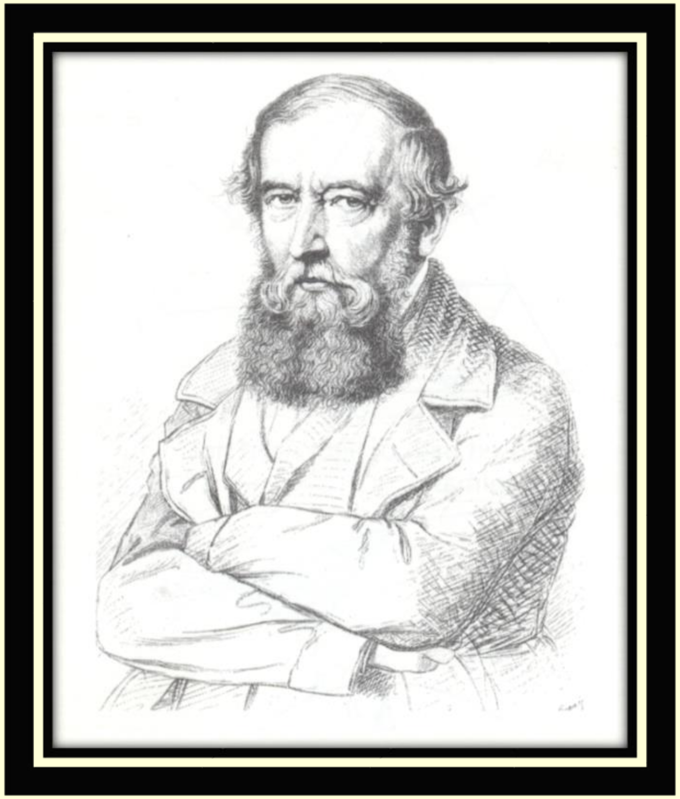
Let $G = (V, E)$ be an arbitrary finite graph.
Fix two real numbers $0 < a < b \leq 2a$.

Define a metric on V as follows: the distance between
adjacent vertices of G equals b, and
non-adjacent vertices of G equals a.

Corollary (A.Ivanov, AT). Let m be the greatest positive integer such that $2d_{GH}(a\Delta_m , V) = b$, then $\gamma(G) = m + 1$
(if there is no such m, then we put $m = 0$).

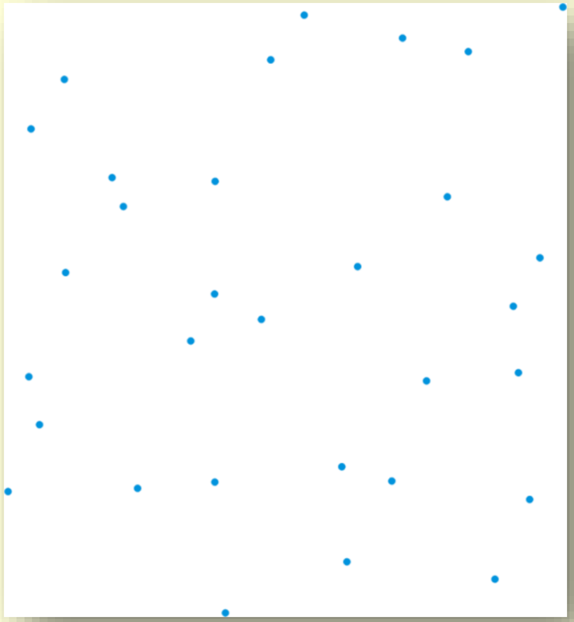
Example. 1) For empty n-graph, $V= a\Delta_n$, and $2\, d_{GH}(a\Delta_m , a\Delta_n) < b$ for all positive m, then $m = 0$ and $\gamma = m + 1 = 1$.
2) Let $G = K_n$, then $V = b\Delta_n$ and $2\, d_{GH}(a\Delta_m , b\Delta_n) < b$ starting from $m = n$, thus $\gamma = n$.
(3 – 5) For the corresponding metric spaces V we obtained exact values for $d_{GH}(a\Delta_m , V)$.

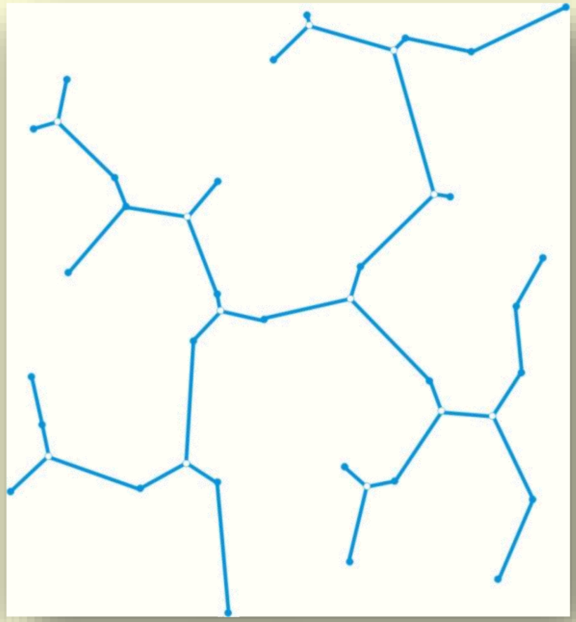
Classical Steiner Problem.



Jacob Steiner
(1796-1863)

Construct a Steiner minimal tree joining a given finite subset of the plane.





Example 3: Steiner Problem on Earth Surface

Steiner Problem on Manhattan plane (Rectilinear Steiner Problem) and chip design

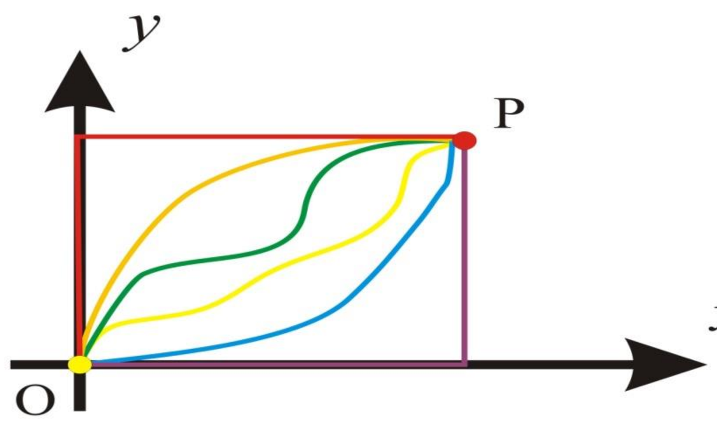
For $A_1 = (x_1, y_1)$ and $A_2 = (x_2, y_2)$

Euclidean distans is

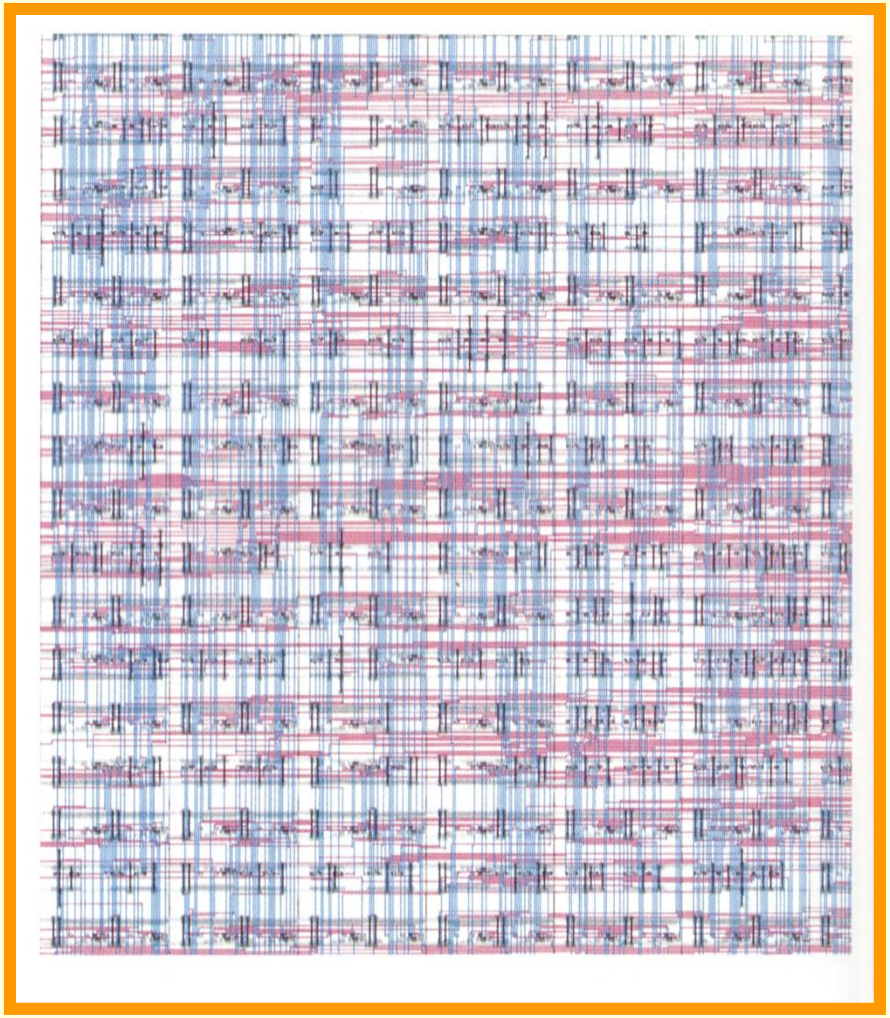
$$\rho_2(A_1, A_2) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

Manhattan distance is

$$\rho_1(A_1, A_2) = |x_1 - x_2| + |y_1 - y_2|$$



All monotonic curves joining O and P have the same Manhattan distance



Example 4: Steiner Problem in Biology

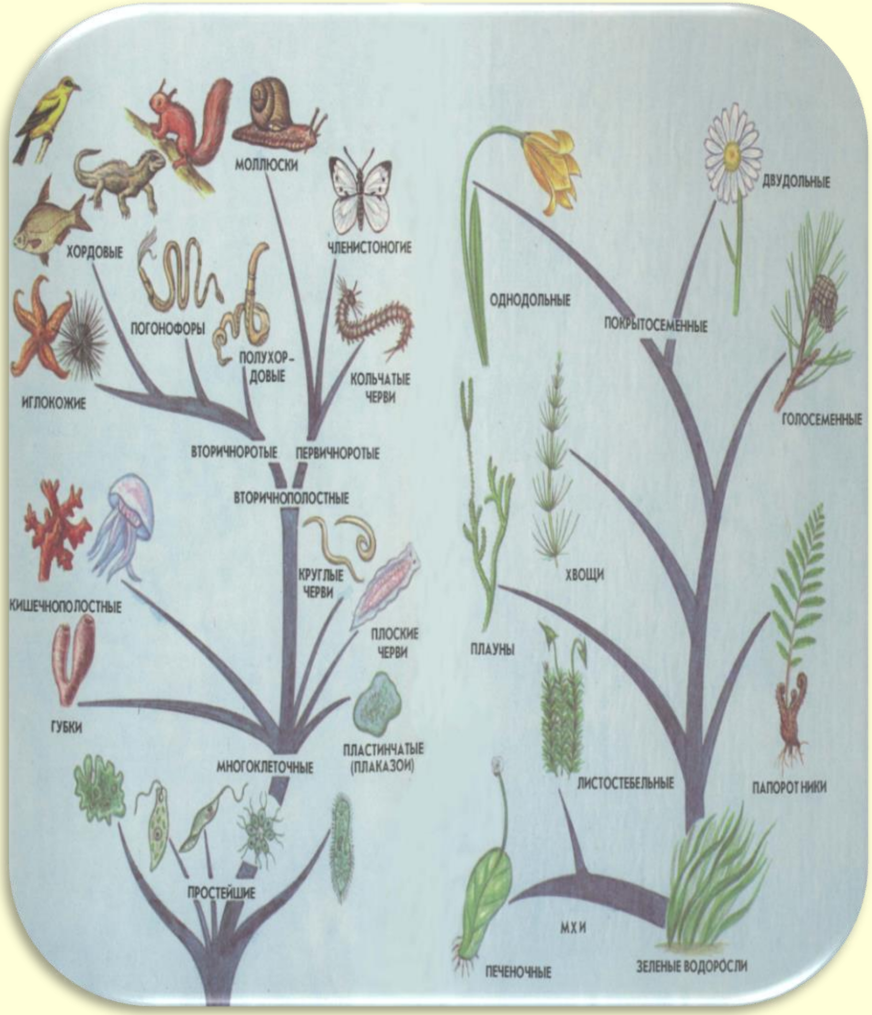
Elementary editor operations on a word are

insertion	**deletion**	**substitution**
$abcd \to ab\mathrm{x}cd$	$ab\mathrm{x}d \to abd$	$ab\mathrm{x}d \to ab\mathrm{y}d$

Hamming distance between two words w_1 and w_2 is the least number of elementary editor operations to pass from w_1 to w_2

To measure the difference between two species one can code them by words, for example, 4-letter DNA word, or 20-letter protein word, or a word characterizing the presence of different phenotypic properties, etc., and to calculate the Hemming distance between these words.

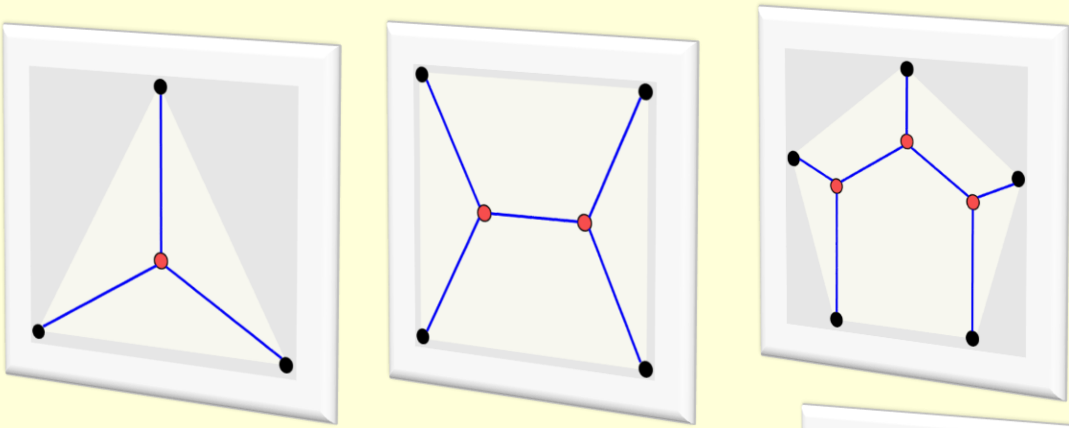
Biological assumption: evolution was optimal in the sense of minimization of the changes number (for example, minimization of mutations number). Thus, the evolution tree has to be the shortest tree (in Hemming distance) joining the words corresponding to nowaday species. Thus enables to reconstruct the properties of predecessors.

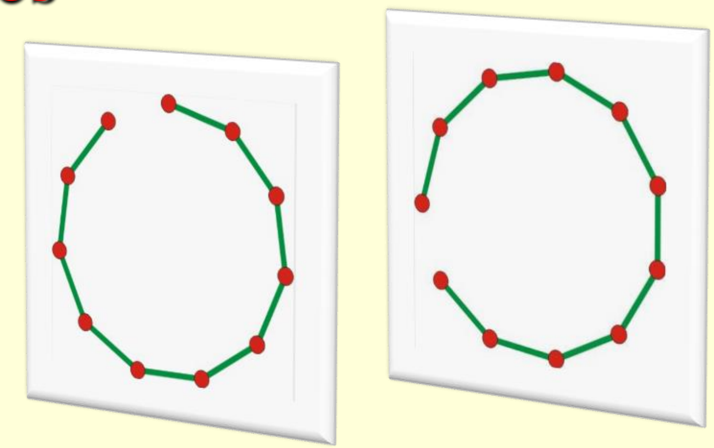


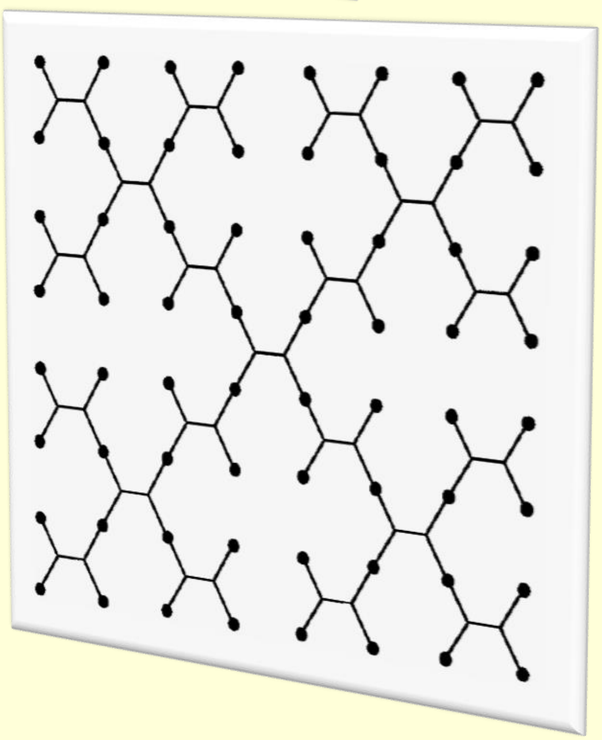
Shortest Trees

Some Examples

Regular polygons:

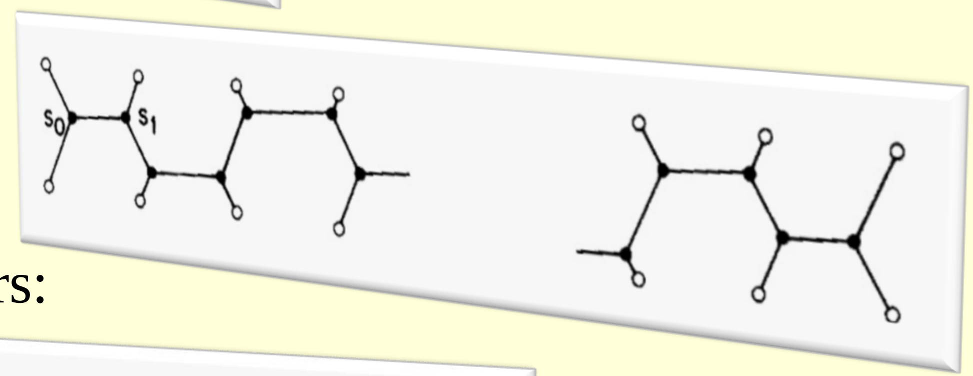


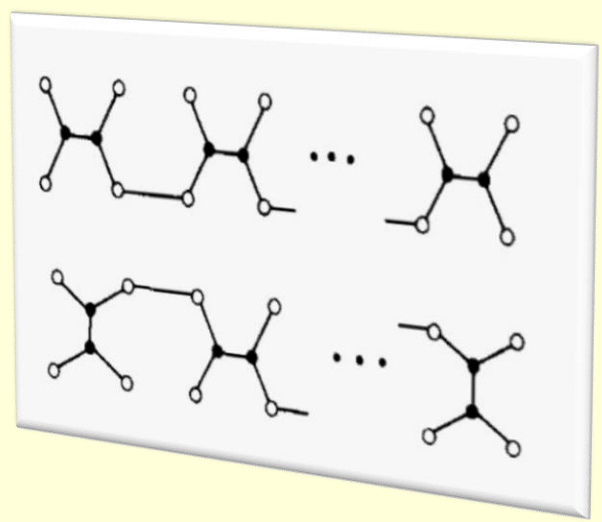




Checkerboard

Ladders:





Zigzag lines,
points on a circle,
etc.

Steiner Problem in $\mathcal{GH}$.

Recall that a (simple) graph G is a pair (V, E), where V and E are classes, and E consists of some 2-point subsets of V. Elements of V are called vertices, and elements of E edges. For convenience reason, we denote the pairs $\{v, w\}$ by vw.

By a walk in G joining vertices v and w, we mean a finite sequence $v_0=v, v_1,\dots,v_k=w$, such that $v_{i-1}v_i \in E$ for all $i = 1,\dots, k$. A graph G is called connected if any two its vertices can be joined by a walk. A walk $v_0, v_1,\dots,v_k$ is called a circle, if $v_0 = v_k$ and all edges $v_{i-1}v_i$ are different. A connected graph without circles is called a tree.

Let $\mathcal{A}$ be a class endowed with an extended pseudometric. We say that $G = (V, E)$ is a graph on $\mathcal{A}$, if $V \subset \mathcal{A}$. For such G, and each $e = vw \in E$, by the length $|e|$ of e we mean $|vw|$. The length $|G|$ of G is $\sum_{e \in E} |e|$, where for an arbitrary class $\{a_i\}_{i\in I}$ of non-negative numbers a_i, we put

$$\sum_{i\in I} a_i = \sup\{\sum_{j\in J} a_j \,,\ J \subset I,\ \#J < \infty\}.$$

Let $M \subset \mathcal{A}$ be a subclass, then we put

$$\mathrm{smt}(M, \mathcal{A}) = \inf\{|G| : G = (V, E) \text{ is a tree on } \mathcal{A}, M \subset V\};$$

If $G = (V, E)$ is a tree on $\mathcal{A}$, $M \subset V$, $|G| = \mathrm{smt}(M, \mathcal{A})$, then G is called a Steiner minimal tree on M. We put

$$\mathrm{SMT}(M, \mathcal{A}) = \{G : G \text{ is a Steiner minimal tree on } M\}.$$

General Steiner Problem in $\mathcal{A}$:

1) describe the subclass $\mathrm{SMT}(\mathcal{A}) = \{M \subset \mathcal{A} : \mathrm{SMT}(M, \mathcal{A}) \neq \emptyset\}$.
2) describe the trees from $\mathrm{SMT}(M, \mathcal{A})$ for various $M \in \mathrm{SMT}(\mathcal{A})$.

Recall that $\mathcal{M}_n = \{X \in \mathcal{M}_n : \#X \leq n\}$.

Theorem (A.Ivanov, N.Nikolaeva, AT).
For any positive integer n and finite $M \subset \mathcal{M}_n$ we have $\mathrm{SMT}(M, \mathcal{M}) \neq \emptyset$.
Moreover, there exists an integer $m = m(n) \geq n$ and a Steiner minimal tree $G = (V, E) \in \mathrm{SMT}(M, \mathcal{M})$ such that $V \subset \mathcal{M}_m$, i.e., $G \in \mathrm{SMT}(M, \mathcal{M}_m)$.

Problem. Is it true that for finite $M \subset \mathcal{GH}$ such that the distances between points of M are finite, it holds $\mathrm{SMT}(M, \mathcal{GH}) \neq \emptyset$?

One-dimensional Gromov minimal fillings

Let M be a class endowed with a generalized pseudometric,

take an arbitrary class X with a generalized pseudometric such that there exists an isometric embedding $f : M \to X$,

$smt(f(M), X)$ is the minimal length of joining trees for $f(M) \subset X$,

$mf(M) = \inf\{s :$ there exists $f : M \to X$ such that $smt(f(M), X) \le s\}$.

Each $G \in SMT(f(M), X)$ such that $|G| = mf(M)$ is called

(one-dimensional Gromov) minimal filling.

Alternative definition:

connected graph $G = (V, E)$ joining M, i.e., $M \subset V$,

weight function $\omega: E \to \mathbb{R}_+$, weighted graph $\mathcal{G} = (V, E, \omega)$,

weight of subgraph $\mathcal{H} \subset \mathcal{G}$ is $\omega(\mathcal{H}) = \sum_{e \in E(\mathcal{H})} \omega(e)$,

distance on V: $d_\omega(x,y) = \inf_\gamma\{\omega(\gamma) \mid \gamma$ is a walk in G joining $x, y \in V\}$.

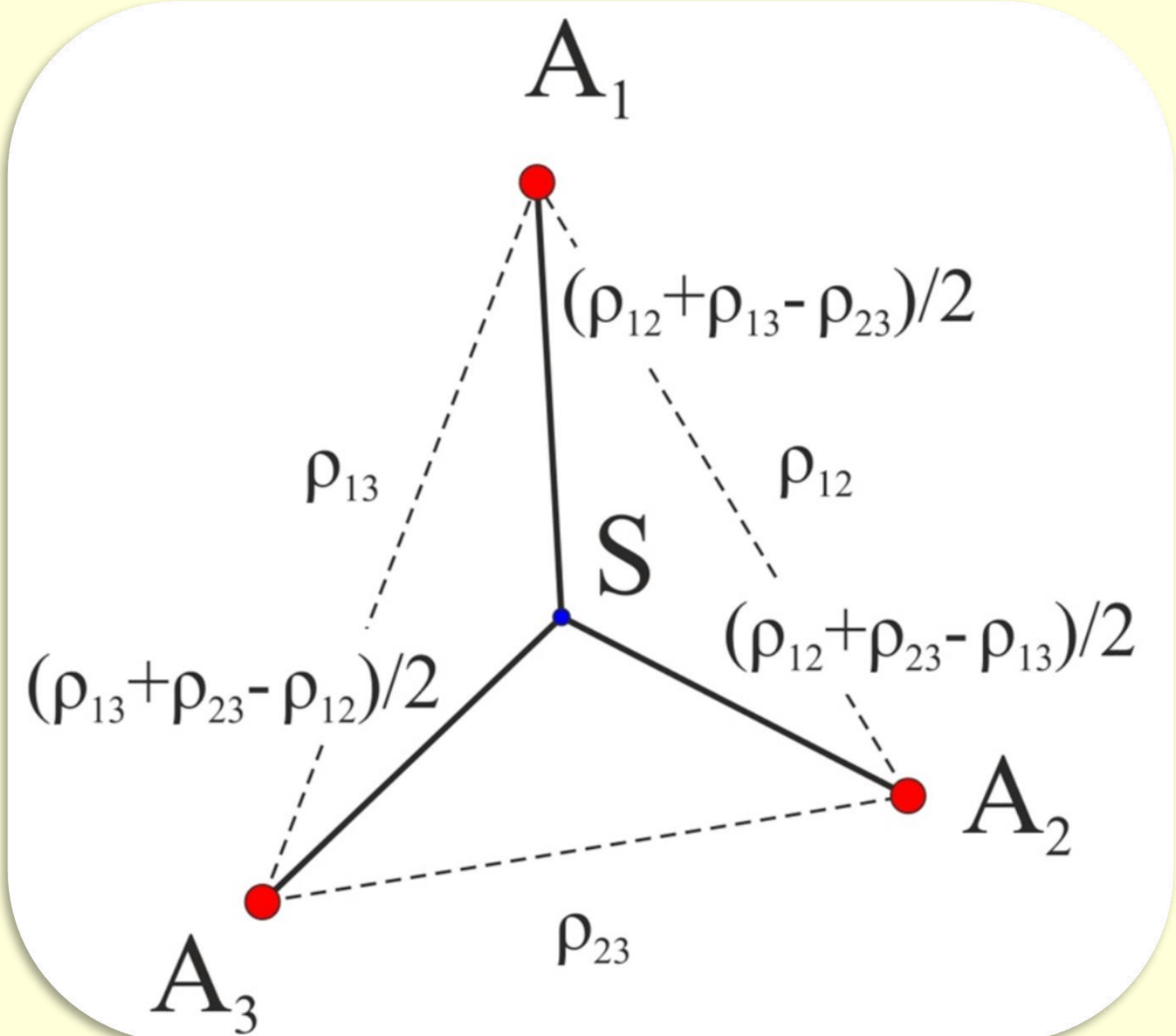
$\mathcal{G}$ is called a filling of M, iff $|xy| \le d_\omega(x,y)$ for all $x, y \in M$,

$mf(M) = \inf\{\omega(\mathcal{G}) : \mathcal{G}$ is a filling of $M\}$,

a filling $\mathcal{G}$ of M such that $\omega(\mathcal{G}) = mf(M)$ is called

(one-dimensional Gromov) minimal filling.

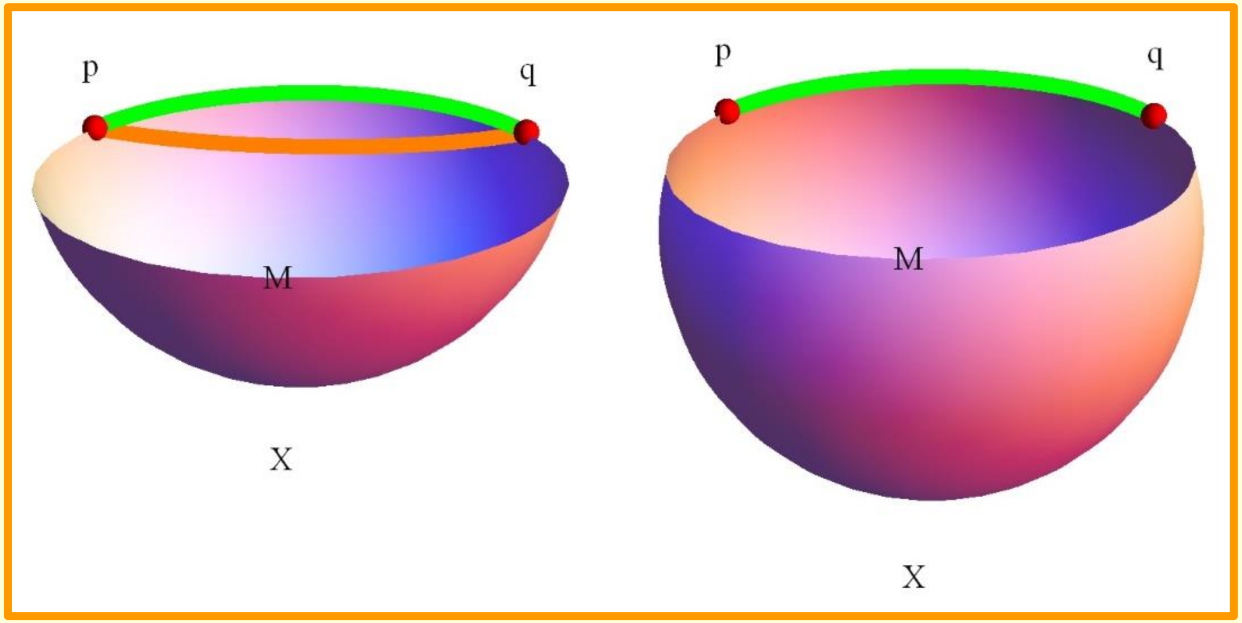
Example: 3-Points Case.



General Gromov minimal fillings.

n-dimensional manifold X with a metric d is called a filling of $(n-1)$-dimensional manifold M with a metric ρ,

if $M = \partial X$, and $\rho(p,q) \le d(p,q)$ for all $p, q \in M$.



Problem (M.Gromov). Given M, find the least possible volume $mf(M)$ of fillings X for M, and describe minimal fillings X, i.e., the ones with

$$mf(M) = volume(X).$$

Minimal fillings, existence, ratios.

Theorem (A.Ivanov, AT).
For any finite metric space M there exists its minimal filling.

Variational curvature of a metric space.

Define Steiner subratio $\mathrm{ssr}(X)$ of a metric space X (Ivanov, Tuzhilin):
for any finite $M \subset X$ we put $\mathrm{ssr}(M, X) = \mathrm{mf}(M)/\mathrm{smt}(M, X)$,
$\mathrm{ssr}(X) = \inf\{\mathrm{ssr}(M, X) : M \subset X, \#M < \infty\}$.
We call a metric space X variationally flat if $\mathrm{ssr}(X) = 1$.

Remark. If a metric space X is variationally flat, then for each finite $M \subset X$ it holds $\mathrm{SMT}(M, X) \neq \emptyset$.

Theorem (Z.Ovsyannikov).
The space $\mathbb{R}^n_\infty$ is variationally flat, i.e., for any finite $M \subset \mathbb{R}^n_\infty$ it holds $\mathrm{SMT}(M, \mathbb{R}^n_\infty) \neq \emptyset$ and any $G \in \mathrm{SMT}(M, \mathbb{R}^n_\infty)$ is a minimal filling for M.

Minimal fillings and Gromov-Hausdorff space.

Recall that $\mathcal{M}_{[n]} = \{X \in \mathcal{M} : \#X = n\}$. One can show that $\mathcal{M}_{[3]}$ is isometric to the cone $C = \{(a, b, c) : 0 < a \le b \le c \le a + b\} \subset \mathbb{R}^n_\infty$.

Is it true that $\mathcal{M}_{[3]}$ and, probably, $\mathcal{M}_{[n]}$ and $\mathcal{M}$ are variationally flat?

Theorem (A.Ivanov, AT). For any positive integer N and any n-points metric space X in general position, $n \ge 2$, there exists a neighborhood $U \subset \mathcal{M}_{[n]}$ of X such that for any $M \subset U$, $\#M \le N$, each Steiner minimal tree on M is a minimal filling for M.

Remark. Perhaps, the condition $\#M \le N$ is redundant.

Example. $M=\{A, B, C\} \subset \mathcal{M}_{[3]}$, where the spaces A, B, C are given by their distances vectors $(8, 22, 29.5)$, $(11.5, 18, 29)$, $(12, 21.5, 33)$, respectively. Then $\mathrm{smt}(M, \mathcal{M}_{[3]}) > \mathrm{mf}(M)$. Moreover, $\mathrm{smt}(M, \mathcal{M}) > \mathrm{mf}(M)$, and, by numerical experiment, $\mathrm{ssr}(\mathcal{M}) \le \mathrm{ssr}(M) < 0.857$.

Problem. Calculate $\mathrm{ssr}(\mathcal{M})$.

MF-universality of Gromov-Hausdorff space.

Theorem (A.Ivanov, AT).
Let X be a finite metric space, and $G=(V, E)$ be its minimal filling (in particular, $X \subset V$). Then there exists a positive integer N and an isometric embedding $f : X \to \mathcal{M}_{[N]}$, $M = f(X)$, such that

(1) there exists an isometric embedding $F : V \to \mathcal{M}_{[N]}$ such that $F|_X = f$;

(2) $F(G) = (F(V), F(E)) \in SMT(M, \mathcal{M})$, i.e., $F(G)$ is a Steiner minimal tree for M;

(3) any Steiner minimal tree in $\mathcal{M}$ joining M is a minimal filling for M.

In other words, any one-dimensional Gromov minimal filling can be realized as a Steiner minimal tree in $\mathcal{M}$. Moreover, one can put this realization into some $\mathcal{M}_{[N]}$.

Thank You!