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8. Higher local skew fields

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n-dimensional local skew fields are a natural generalization of n-dimensional local fields. The latter have numerous applications to problems of algebraic geometry, both arithmetical and geometrical, as it is shown in this volume. From this viewpoint, it would be reasonable to restrict oneself to commutative fields only. Nevertheless, already in class field theory one meets non-commutative rings which are skew fields finite-dimensional over their center K. For example, K is a (commutative) local field and the skew field represents elements of the Brauer group of the field K (see also an example below). In [Pa] A.N. Parshin pointed out another class of non-commutative local fields arising in differential equations and showed that these skew fields possess many features of commutative fields. He defined a skew field of formal pseudo-differential operators in n variables and studied some of their properties. He raised a problem of classifying non-commutative local skew fields.

In this section we treat the case of $n = 2$ and list a number of results, in particular a classification of certain types of 2-dimensional local skew fields.

8.1. Basic definitions

Definition. A skew field K is called a *complete discrete valuation skew field* if K is complete with respect to a discrete valuation (the residue skew field is not necessarily commutative). A field K is called an *n-dimensional local skew field* if there are skew fields $K = K_n, K_{n-1}, \ldots, K_0$ such that each K_i for $i > 0$ is a complete discrete valuation skew field with residue skew field K_{i-1}.

Examples.

(1) Let k be a field. Formal pseudo-differential operators over $k((X))$ form a 2-dimensional local skew field $K = k((X))((\partial_X^{-1}))$, $\partial_X X = X\partial_X + 1$. If $\mathrm{char}(k) = 0$ we get an example of a skew field which is an infinite dimensional vector space over its centre.

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(2) Let L be a local field of equal characteristic (of any dimension). Then an element of $\mathrm{Br}(L)$ is an example of a skew field which is finite dimensional over its centre.

From now on *let* K be a two-dimensional local skew field. Let t_2 be a generator of $\mathcal{M}_{K_2}$ and t_1' be a generator of $\mathcal{M}_{K_1}$. If $t_1 \in K$ is a lifting of t_1' then t_1, t_2 is called a *system of local parameters* of K. We denote by v_{K_2} and v_{K_1} the (surjective) discrete valuations of K_2 and K_1 associated with t_2 and t_1'.

Definition. A two-dimensional local skew field K is said to *split* if there is a section of the homomorphism $\mathcal{O}_{K_2} \to K_1$ where $\mathcal{O}_{K_2}$ is the ring of integers of K_2.

Example (N. Dubrovin). Let $\mathbb{Q}((u))\langle x, y\rangle$ be a free associative algebra over $\mathbb{Q}((u))$ with generators x, y. Let $I = \langle [x,[x,y]], [y,[x,y]]\rangle$. Then the quotient

$$A = \mathbb{Q}((u))\langle x, y\rangle / I$$

is a $\mathbb{Q}$-algebra which has no non-trivial zero divisors, and in which $z = [x,y] + I$ is a central element. Any element of A can be uniquely represented in the form

$$f_0 + f_1 z + \ldots + f_m z^m$$

where $f_0, \ldots, f_m$ are polynomials in the variables x, y.

One can define a discrete valuation w on A such that $w(x) = w(y) = w(\mathbb{Q}((u))) = 0$, $w([x,y]) = 1$, $w(a) = k$ if $a = f_k z^k + \ldots + f_m z^m$, $f_k \neq 0$. The skew field B of fractions of A has a discrete valuation v which is a unique extension of w. The completion of B with respect to v is a two-dimensional local skew field which does not split (for details see [Zh, Lemma 9]).

Definition. Assume that K_1 is a field. The homomorphism

$$\varphi_0 \colon K^* \to \mathrm{Int}(K), \quad \varphi_0(x)(y) = x^{-1} y x$$

induces a homomorphism $\varphi \colon K_2^* / \mathcal{O}_{K_2}^* \to \mathrm{Aut}(K_1)$. The *canonical automorphism* of K_1 is $\alpha = \varphi(t_2)$ where t_2 is an arbitrary prime element of K_2.

Definition. Two two-dimensional local skew fields K and K' are *isomorphic* if there is an isomorphism $K \to K'$ which maps $\mathcal{O}_K$ onto $\mathcal{O}_{K'}$, $\mathcal{M}_K$ onto $\mathcal{M}_{K'}$ and $\mathcal{O}_{K_1}$ onto $\mathcal{O}_{K_1'}$, $\mathcal{M}_{K_1}$ onto $\mathcal{M}_{K_1'}$.

8.2. Canonical automorphisms of infinite order

Theorem.

(1) *Let K be a two-dimensional local skew field. If $\alpha^n \neq \mathrm{id}$ for all $n \geqslant 1$ then* $\mathrm{char}(K_2) = \mathrm{char}(K_1)$, *$K$ splits and K is isomorphic to a two-dimensional local skew field $K_1((t_2))$ where $t_2 a = \alpha(a) t_2$ for all $a \in K_1$.*

(2) *Let K, K' be two-dimensional local skew fields and let K_1, K_1' be fields. Let $\alpha^n \neq \mathrm{id}$, $\alpha'^n \neq \mathrm{id}$ for all $n \geqslant 1$. Then K is isomorphic to K' if and only if there is an isomorphism $f\colon K_1 \to K_1'$ such that $\alpha = f^{-1}\alpha' f$ where α, α' are the canonical automorphisms of K_1 and K_1'.*

Remarks.

1. This theorem is true for any higher local skew field.
2. There are examples (similar to Dubrovin's example) of local skew fields which do not split and in which $\alpha^n = \mathrm{id}$ for some positive integer n.

Proof. (2) follows from (1). We sketch the proof of (1). For details see [Zh, Th.1].

If $\mathrm{char}(K) \neq \mathrm{char}(K_1)$ then $\mathrm{char}(K_1) = p > 0$. Hence $v(p) = r > 0$. Then for any element $t \in K$ with $v(t) = 0$ we have $ptp^{-1} \equiv \alpha^r(\bar t) \bmod \mathcal{M}_K$ where $\bar t$ is the image of t in K_1. But on the other hand, $pt = tp$, a contradiction.

Let F be the prime field in K. Since $\mathrm{char}(K) = \mathrm{char}(K_1)$ the field F is a subring of $\mathcal{O} = \mathcal{O}_{K_2}$. One can easily show that there exists an element $c \in K_1$ such that $\alpha^n(c) \neq c$ for every $n \geqslant 1$ [Zh, Lemma 5].

Then any lifting c' in $\mathcal{O}$ of c is transcendental over F. Hence we can embed the field $F(c')$ in $\mathcal{O}$. Let $\bar L$ be a maximal field extension of $F(c')$ which can be embedded in $\mathcal{O}$. Denote by L its image in $\mathcal{O}$. Take $\bar a \in K_1 \setminus \bar L$. We claim that there exists a lifting $a' \in \mathcal{O}$ of $\bar a$ such that a' commutes with every element in L. To prove this fact we use the completeness of $\mathcal{O}$ in the following argument.

Take any lifting a in $\mathcal{O}$ of $\bar a$. For every element $x \in L$ we have $axa^{-1} \equiv x \bmod \mathcal{M}_K$. If t_2 is a prime element of K_2 we can write

$$axa^{-1} = x + \delta_1(x) t_2$$

where $\delta_1(x) \in \mathcal{O}$. The map $\overline{\delta_1}\colon L \ni x \to \overline{\delta_1(x)} \in K_1$ is an α-derivation, i.e.

$$\overline{\delta_1}(ef) = \overline{\delta_1}(e)\alpha(f) + e\overline{\delta_1}(f)$$

for all $e, f \in L$. Take an element h such that $\alpha(h) \neq h$, then $\overline{\delta_1}(a) = g\alpha(a) - ag$ where $g = \overline{\delta_1}(h)/(\alpha(h) - h)$. Therefore there is $a_1 \in K_1$ such that

$$(1 + a_1 t_2) axa^{-1} (1 + a_1 t_2)^{-1} \equiv x \bmod \mathcal{M}_K^2.$$

By induction we can find an element $a' = \ldots \cdot (1 + a_1 t_2) a$ such that $a' x a'^{-1} = x$.

Now, if $\overline{a}$ is not algebraic over $\overline{L}$, then for its lifting $a' \in \mathcal{O}$ which commutes with L we would deduce that $L(a')$ is a field extension of $F(c')$ which can be embedded in $\mathcal{O}$, which contradicts the maximality of L.

Hence $\overline{a}$ is algebraic and separable over $\overline{L}$. Using a generalization of Hensel's Lemma [Zh, Prop.4] we can find a lifting a' of $\overline{a}$ such that a' commutes with elements of L and a' is algebraic over L, which again leads to a contradiction.

Finally let $\overline{a}$ be purely inseparable over $\overline{L}$, $\overline{a}^{p^k} = \overline{x}$, $x \in L$. Let a' be its lifting which commutes with every element of L. Then ${a'}^{p^k} - x$ commutes with every element of L. If $v_K({a'}^{p^k} - x) = r \neq \infty$ then similarly to the beginning of this proof we deduce that the image of $({a'}^{p^k} - x)c({a'}^{p^k} - x)^{-1}$ in K_1 is equal to $\alpha^r(c)$ (which is distinct from c), a contradiction. Therefore, ${a'}^{p^k} = x$ and the field $L(a')$ is a field extension of $F(c')$ which can be embedded in $\mathcal{O}$, which contradicts the maximality of L.

Thus, $\overline{L} = K_1$.

To prove that K is isomorphic to a skew field $K_1((t_2))$ where $t_2 a = \alpha(a) t_2$ one can apply similar arguments as in the proof of the existence of an element a' such that $a' x {a'}^{-1} = x$ (see above). So, one can find a parameter t_2 with a given property. □

In some cases we have a complete classification of local skew fields.

Proposition ([Zh]). *Assume that K_1 is isomorphic to $k((t_1))$. Put*

$$\zeta = \alpha(t_1) t_1^{-1} \bmod \mathcal{M}_{K_1}.$$

Put $i_\alpha = 1$ if ζ is not a root of unity in k and $i_\alpha = v_{K_1}(\alpha^n(t_1) - t_1)$ if ζ is a primitive nth root. Assume that k is of characteristic zero. Then there is an automorphism $f \in \mathrm{Aut}_k(K_1)$ such that $f^{-1}\alpha f = \beta$ where

$$\beta(t_1) = \zeta t_1 + x t_1^{i_\alpha} + x^2 y t_1^{2i_\alpha - 1}$$

for some $x \in k^*/k^{*(i_\alpha - 1)}$, $y \in k$.

Two automorphisms α and β are conjugate if and only if

$$(\zeta(\alpha), i_\alpha, x(\alpha), y(\alpha)) = (\zeta(\beta), i_\beta, x(\beta), y(\beta)).$$

Proof. First we prove that $\alpha = f \beta' f^{-1}$ where

$$\beta'(t_1) = \zeta t_1 + x t_1^{in+1} + y t_1^{2in+1}$$

for some natural i. Then we prove that $i_\alpha = i_{\beta'}$.

Consider a set $\{\alpha_i : i \in \mathbb{N}\}$ where $\alpha_i = f_i \alpha_{i-1} f_i^{-1}$, $f_i(t_1) = t_1 + x_i t_1^i$ for some $x_i \in k$, $\alpha_1 = \alpha$. Write

$$\alpha_i(t_1) = \zeta t_1 + a_{2,i} t_1^2 + a_{3,i} t_1^3 + \dots .$$

One can check that $a_{2,2} = x_2(\zeta^2 - \zeta) + a_{2,1}$ and hence there exists an element $x_2 \in k$ such that $a_{2,2} = 0$. Since $a_{j,i+1} = a_{j,i}$, we have $a_{2,j} = 0$ for all $j \geqslant 2$. Further, $a_{3,3} = x_3(\zeta^3 - \zeta) + a_{3,2}$ and hence there exists an element $x_3 \in k$ such that $a_{3,3} = 0$. Then $a_{3,j} = 0$ for all $j \geqslant 3$. Thus, any element $a_{k,k}$ can be made equal to zero if $n \not| (k-1)$, and therefore $\alpha = f\tilde{\alpha}f^{-1}$ where

$$\tilde{\alpha}(t_1) = \zeta t_1 + \tilde{a}_{in+1}t_1^{in+1} + \tilde{a}_{in+n+1}t_1^{in+n+1} + \dots$$

for some i, $\tilde{a}_j \in k$. Notice that $\tilde{a}_{in+1}$ does not depend on x_i. Put $x = x(\alpha) = \tilde{a}_{in+1}$.

Now we replace α by $\tilde{\alpha}$. One can check that if $n | (k-1)$ then

$$a_{j,k} = a_{j,k-1} \qquad \text{for } 2 \leqslant j < k + in$$

and

$$a_{k+in,k} = x_k x(k - in - 1) + a_{k+in} + \text{ some polynomial which does not depend on } x_k.$$

From this fact it immediately follows that $a_{2in+1,in+1}$ does not depend on x_i and for all $k \neq in+1$ $a_{k+in,k}$ can be made equal to zero. Then $y = y(\alpha) = a_{2in+1,in+1}$.

Now we prove that $i_\alpha = i_{\beta'}$. Using the formula

$$\beta'^n(t_1) = t_1 + nx(\alpha)\zeta^{-1}t_1^{in+1} + \dots$$

we get $i_{\beta'} = in+1$. Then one can check that $v_{K_1}(f^{-1}(\alpha^n - \mathrm{id}\,)f) = v_{K_1}(\alpha^n - \mathrm{id}\,) = i_\alpha$. Since $\beta'^n - \mathrm{id} = f^{-1}(\alpha^n - \mathrm{id}\,)f$, we get the identity $i_\alpha = i_{\beta'}$.

The rest of the proof is clear. For details see [Zh, Lemma 6 and Prop.5]. □

8.3. Canonical automorphisms of finite order

8.3.1. Characteristic zero case.

Assume that

a two-dimensional local skew field K splits,
K_1 is a field, $K_0 \subset Z(K)$,
$\mathrm{char}\,(K) = \mathrm{char}\,(K_0) = 0$,
$\alpha^n = \mathrm{id}$ for some $n \geqslant 1$,
for any convergent sequence (a_j) in K_1 the sequence $(t_2 a_j t_2^{-1})$ converges in K.

Lemma. *K is isomorphic to a two-dimensional local skew field $K_1((t_2))$ where*

$$t_2 a t_2^{-1} = \alpha(a) + \delta_i(a)t_2^i + \delta_{2i}(a)t_2^{2i} + \delta_{2i+n}(a)t_2^{2i+n} + \dots \qquad \textit{for all } a \in K_1$$

where $n | i$ and $\delta_j : K_1 \to K_1$ are linear maps and

$$\delta_i(ab) = \delta_i(a)\alpha(b) + \alpha(a)\delta_i(b) \qquad \textit{for every } a, b \in K_1 .$$

Moreover

$$t_2^n a t_2^{-n} = a + \delta'_i(a)t_2^i + \delta'_{2i}(a)t_2^{2i} + \delta'_{2i+n}(a)t_2^{2i+n} + \dots$$

where δ'_j *are linear maps and* δ'_i *and* $\delta := \delta'_{2i} - ((i+1)/2){\delta'_i}^2$ *are derivations.*

Remark. The following fact holds for the field K of any characteristic: K is isomorphic to a two-dimensional local skew field $K_1((t_2))$ where

$$t_2 a t_2^{-1} = \alpha(a) + \delta_i(a)t_2^i + \delta_{i+1}(a)t_2^{i+1} + \dots$$

where δ_j are linear maps which satisfy some identity. For explicit formulas see [Zh, Prop.2 and Cor.1].

Proof. It is clear that K is isomorphic to a two-dimensional local skew field $K_1((t_2))$ where

$$t_2 a t_2^{-1} = \alpha(a) + \delta_1(a)t_2 + \delta_2(a)t_2^2 + \dots \qquad \text{for all } a$$

and δ_j are linear maps. Then δ_1 is a (α^2, α)-derivation, that is $\delta_1(ab) = \delta_1(a)\alpha^2(b) + \alpha(a)\delta_1(b)$.

Indeed,

$$t_2 ab t_2^{-1} = t_2 a t_2^{-1} t_2 b t_2^{-1} = (\alpha(a) + \delta_1(a)t_2 + \dots)(\alpha(b) + \delta_1(b)t_2 + \dots)$$
$$= \alpha(a)\alpha(b) + (\delta_1(a)\alpha^2(b) + \alpha(a)\delta_1(b))t_2 + \dots = \alpha(ab) + \delta_1(ab)t_2 + \dots .$$

From the proof of Theorem 8.2 it follows that δ_1 is an inner derivation, i.e. $\delta_1(a) = g\alpha^2(a) - \alpha(a)g$ for some $g \in K_1$, and that there exists a $t_{2,2} = (1 + x_1 t_2)t_2$ such that

$$t_{2,2} a t_{2,2}^{-1} = \alpha(a) + \delta_{2,2}(a)t_{2,2}^2 + \dots .$$

One can easily check that $\delta_{2,2}$ is a (α^3, α)-derivation. Then it is an inner derivation and there exists $t_{2,3}$ such that

$$t_{2,3} a t_{2,3}^{-1} = \alpha(a) + \delta_{3,3}(a)t_{2,3}^3 + \dots .$$

By induction one deduces that if

$$t_{2,j} a t_{2,j}^{-1} = \alpha(a) + \delta_{n,j}(a)t_{2,j}^n + \dots + \delta_{kn,j}(a)t_{2,j}^{kn} + \delta_{j,j}(a)t_{2,j}^j + \dots$$

then $\delta_{j,j}$ is a (α^{j+1}, α)-derivation and there exists $t_{2,j+1}$ such that

$$t_{2,j+1} a t_{2,j+1}^{-1} = \alpha(a) + \delta_{n,j}(a)t_{2,j+1}^n + \dots + \delta_{kn,j}(a)t_{2,j+1}^{kn} + \delta_{j+1,j+1}(a)t_{2,j+1}^{j+1} + \dots .$$

The rest of the proof is clear. For details see [Zh, Prop.2, Cor.1, Lemmas 10, 3]. □

Definition. Let $i = v_{K_2}(\varphi(t_2^n)(t_1) - t_1) \in n\mathbb{N} \cup \infty$, ($\varphi$ is defined in subsection 8.1) and let $r \in \mathbb{Z}/i$ be $v_{K_1}(x) \bmod i$ where x is the residue of $(\varphi(t_2^n)(t_1) - t_1)t_2^{-i}$. Put

$$a = \operatorname{res}_{t_1}\left(\frac{(\delta'_{2i} - \frac{i+1}{2}{\delta'_i}^2)(t_1)}{\delta'_i(t_1)^2} dt_1\right) \in K_0.$$

(δ'_i, δ'_{2i} are the maps from the preceding lemma).

Proposition. *If* $n = 1$ *then* i, r *don't depend on the choice of a system of local parameters*; *if* $i = 1$ *then* a *does not depend on the choice of a system of local parameters*; *if* $n \neq 1$ *then* a *depends only on the maps* $\delta_{i+1}, \dots, \delta_{2i-1}$, i, r *depend only on the maps* δ_j, $j \notin n\mathbb{N}$, $j < i$.

Proof. We comment on the statement first. The maps δ_j are uniquely defined by parameters t_1, t_2 and they depend on the choice of these parameters. So the claim that i, r depend only on the maps δ_j, $j \notin n\mathbb{N}$, $j < i$ means that i, r don't depend on the choice of parameters t_1, t_2 which preserve the maps δ_j, $j \notin n\mathbb{N}$, $j < i$.

Note that r depends only on i. Hence it is sufficient to prove the proposition only for i and a. Moreover it suffices to prove it for the case where $n \neq 1$, $i \neq 1$, because if $n = 1$ then the sets $\{\delta_j : j \notin n\mathbb{N}\}$ and $\{\delta_{i+1} : \dots, \delta_{2i-1}\}$ are empty.

It is clear that i depends on δ_j, $j \notin n\mathbb{N}$. Indeed, it is known that δ_1 is an inner (α^2, α)-derivation (see the proof of the lemma). By [Zh, Lemma 3] we can change a parameter t_2 such that δ_1 can be made equal $\delta_1(t_1) = t_1$. Then one can see that $i = 1$. From the other hand we can change a parameter t_2 such that δ_1 can be made equal to 0. In this case $i > 1$. This means that i depends on δ_1. By [Zh, Cor.3] any map δ_j is uniquely determined by the maps δ_q, $q < j$ and by an element $\delta_j(t_1)$. Then using similar arguments and induction one deduces that i depends on other maps δ_j, $j \notin n\mathbb{N}$, $j < i$.

Now we prove that i does not depend on the choice of parameters t_1, t_2 which preserve the maps δ_j, $j \notin n\mathbb{N}$, $j < i$.

Note that i does not depend on the choice of t_1: indeed, if $t'_1 = t_1 + bz^j$, $b \in K_1$ then $z^n t'_1 z^{-n} = z^n t_1 z^{-n} + (z^n b z^{-n}) z^j = t'_1 + r$, where $r \in \mathcal{M}_K^i \setminus \mathcal{M}_K^{i+1}$. One can see that the same is true for $t'_1 = c_1 t_1 + c_2 t_2^2 + \dots$, $c_j \in K_0$.

Let δ_q be the first non-zero map for given t_1, t_2. If $q \neq i$ then by [Zh, Lemma 8, (ii)] there exists a parameter t'_1 such that $z t'_1 z^{-1} = {t'_1}^{\alpha} + \delta_{q+1}(t'_1) z^{q+1} + \dots$. Using this fact and Proposition 8.2 we can reduce the proof to the case where $q = i$, $\alpha(t_1) = \xi t_1$, $\alpha(\delta_i(t_1)) = \xi \delta_i(t_1)$ (this case is equivalent to the case of $n = 1$). Then we apply [Zh, Lemma 3] to show that

$$v_{K_2}((\phi(t'_2) - 1)(t_1)) = v_{K_2}((\phi(t_2) - 1)(t_1)),$$

for any parameters t_2, t'_2, i.e. i does not depend on the choice of a parameter t_2. For details see [Zh, Prop.6].

To prove that a depends only on $\delta_{i+1}, \dots, \delta_{2i-1}$ we use the fact that for any pair of parameters t'_1, t'_2 we can find parameters $t''_1 = t_1 + r$, where $r \in \mathcal{M}_K^i$, t''_2 such that corresponding maps δ_j are equal for all j. Then by [Zh, Lemma 8] a does not depend on t''_1 and by [Zh, Lemma 3] a depends on $t''_2 = t_2 + a_1 t_2^2 + \dots$, $a_j \in K_1$ if and only if $a_1 = \dots = a_{i-1}$. Using direct calculations one can check that a doesn't depend on $t''_2 = a_0 t_2$, $a_0 \in K_1^*$.

To prove the fact it is sufficient to prove it for $t_1'' = t_1 + ct_1^h z^j$ for any $j < i$, $c \in K_0$. Using [Zh, Lemma 8] one can reduce the proof to the assertion that some identity holds. The identity is, in fact, some equation on residue elements. One can check it by direct calculations. For details see [Zh, Prop.7]. □

Remark. The numbers i, r, a can be defined only for local skew fields which splits. One can check that the definition can not be extended to the skew field in Dubrovin's example.

Theorem.

(1) K *is isomorphic to a two-dimensional local skew field* $K_0((t_1))((t_2))$ *such that*

$$t_2 t_1 t_2^{-1} = \xi t_1 + x t_2^i + y t_2^{2i}$$

where ξ *is a primitive* n *th root,* $x = ct_1^r$, $c \in K_0^*/(K_0^*)^d$,

$$y = (a + r(i+1)/2) t_1^{-1} x^2, \quad d = \gcd(r-1, i).$$

If $n = 1$, $i = \infty$, *then* K *is a field.*

(2) *Let* K, K' *be two-dimensional local skew fields of characteristic zero which splits; and let* K_1, K_1' *be fields. Let* $\alpha^n = \mathrm{id}$, $\alpha'^{n'} = \mathrm{id}$ *for some* $n, n' \geqslant 1$. *Then* K *is isomorphic to* K' *if and only if* K_0 *is isomorphic to* K_0' *and the ordered sets* (n, ξ, i, r, c, a) *and* $(n', \xi', i', r', c', a')$ *coincide.*

Proof. (2) follows from the Proposition of 8.2 and (1). We sketch the proof of (1).

From Proposition 8.2 it follows that there exists t_1 such that $\alpha(t_1) = \xi t_1$; $\delta_i(t_1)$ can be represented as $ct_1^r a^i$. Hence there exists t_2 such that

$$t_2 t_1 t_2^{-1} = \xi t_1 + x t_2^i + \delta_{2i}(t_1) t_2^{2i} + \dots$$

Using [Zh, Lemma 8] we can find a parameter $t_1' = t_1 \bmod \mathcal{M}_K$ such that

$$t_2 t_1' t_2^{-1} = \xi t_1 + x t_2^i + y t_2^{2i} + \dots$$

The rest of the proof is similar to the proof of the lemma. Using [Zh, Lemma 3] one can find a parameter $t_2' = t_2 \bmod \mathcal{M}_K^2$ such that $\delta_j(t_1) = 0$, $j > 2i$. □

Corollary. *Every two-dimensional local skew field* K *with the ordered set*

$$(n, \xi, i, r, c, a)$$

is a finite-dimensional extension of a skew field with the ordered set $(1, 1, 1, 0, 1, a)$.

Remark. There is a construction of a two-dimensional local skew field with a given set (n, ξ, i, r, c, a).

Examples.

(1) The ring of formal pseudo-differential equations is the skew field with the set $(n = 1, \xi = 1, i = 1, r = 0, c = 1, a = 0)$.

(2) The elements of $\mathrm{Br}(L)$ where L is a two-dimensional local field of equal characteristic are local skew fields. If, for example, L is a C_2- field, they split and $i = \infty$. Hence any division algebra in $\mathrm{Br}(L)$ is cyclic.

8.3.2. Characteristic p case.

Theorem. *Suppose that a two-dimensional local skew field K splits, K_1 is a field, $K_0 \subset Z(K)$, $\mathrm{char}\,(K) = \mathrm{char}\,(K_0) = p > 2$ and $\alpha = \mathrm{id}$.*

Then K is a finite dimensional vector space over its center if and only if K is isomorphic to a two-dimensional local skew field $K_0((t_1))((t_2))$ where

$$t_2^{-1} t_1 t_2 = t_1 + x t_2^i$$

with $x \in K_1^p$, $(i, p) = 1$.

Proof. The "if" part is obvious. We sketch the proof of the "only if" part.

If K is a finite dimensional vector space over its center then K is a division algebra over a henselian field. In fact, the center of K is a two-dimensional local field $k((u))((t))$. Then by [JW, Prop.1.7] $K_1/(Z(K))_1$ is a purely inseparable extension. Hence there exists t_1 such that $t_1^{p^k} \in Z(K)$ for some $k \in \mathbb{N}$ and $K \simeq K_0((t_1))((t_2))$ as a vector space with the relation

$$t_2 t_1 t_2^{-1} = t_1 + \delta_i(t_1) t_2^i + \dots$$

(see Remark 8.3.1). Then it is sufficient to show that i is prime to p and there exist parameters $t_1 \in K_1$, t_2 such that the maps δ_j satisfy the following property:

(*) If j is not divisible by i then $\delta_j = 0$. If j is divisible by i then $\delta_j = c_{j/i} \delta_i^{j/i}$ with some $c_{j/i} \in K_1$.

Indeed, if this property holds then by induction one deduces that $c_{j/i} \in K_0$, $c_{j/i} = ((i+1) \dots (i(j/i - 1) + 1))/(j/i)!$. Then one can find a parameter $t_2' = b t_2$, $b \in K_1$ such that δ_j' satisfies the same property and $\delta_i^2 = 0$. Then

$$t_2'^{\,-1} t_1 t_2' = t_1 - \delta_i'(t_1) t_2^i.$$

First we prove that $(i, p) = 1$. To show it we prove that if $p|i$ then there exists a map δ_j such that $\delta_j(t_1^{p^k}) \neq 0$. To find this map one can use [Zh, Cor.1] to show that $\delta_{ip}(t_1^p) \neq 0$, $\delta_{ip^2}(t_1^{p^2}) \neq 0$, $\dots$, $\delta_{ip^k}(t_1^{p^k}) \neq 0$.

Then we prove that for some t_2 property (*) holds. To show it we prove that if property (*) does not hold then there exists a map δ_j such that $\delta_j(t_1^{p^k}) \neq 0$. To find this map we reduce the proof to the case of $i \equiv 1 \bmod p$. Then we apply the following idea.

Let $j \equiv 1 \bmod p$ be the minimal positive integer such that δ_j is not equal to zero on $K_1^{p^l}$. Then one can prove that the maps $\delta_m,\ kj \leqslant m < (k+1)j,\ k \in \{1, \dots, p-1\}$ satisfy the following property:

there exist elements $c_{m,k} \in K_1$ such that

$$(\delta_m - c_{m,1}\delta - \dots - c_{m,k}\delta^k)|_{K_1^{p^l}} = 0$$

where $\delta\colon K_1 \to K_1$ is a linear map, $\delta|_{K_1^{p^l}}$ is a derivation, $\delta(t_1^j) = 0$ for $j \notin p^l\mathbb{N}$, $\delta(t_1^{p^l}) = 1,\ c_{kj,k} = c(\delta_j(t_1^{p^l}))^k,\ c \in K_0$.

Now consider maps $\widetilde{\delta_q}$ which are defined by the following formula

$$t_2^{-1}at_2 = a + \widetilde{\delta_i}(a)t_2^i + \widetilde{\delta_{i+1}}(a)t_2^{i+1} + \dots, \quad a \in K_1.$$

Then $\widetilde{\delta_q} + \delta_q + \sum_{k=1}^{q-1} \delta_k \widetilde{\delta_{q-k}} = 0$ for any q. In fact, $\widetilde{\delta_q}$ satisfy some identity which is similar to the identity in [Zh, Cor.1]. Using that identity one can deduce that
if

$j \equiv 1 \bmod p$ and there exists the minimal m ($m \in \mathbb{Z}$) such that $\delta_{mp+2i}|_{K_1^{p^l}} \neq 0$ if $j \nmid (mp+2i)$ and $\delta_{mp+2i}|_{K_1^{p^l}} \neq s\delta_j^{(2i+mp)/j}|_{K_1^{p^l}}$ for any $s \in K_1$ otherwise, and $\delta_q(t_1^{p^l}) = 0$ for $q < mp+2i,\ q \not\equiv 1 \bmod p$,

then

$(mp+2i)+(p-1)j$ is the minimal integer such that $\delta_{(mp+2i)+(p-1)j}|_{K_1^{p^{l+1}}} \neq 0$.

To complete the proof we use induction and [Zh, Lemma 3] to show that there exist parameters $t_1 \in K_1,\ t_2$ such that $\delta_q(t_1^{p^l}) = 0$ for $q \not\equiv 1, 2 \bmod p$ and $\delta_j^2 = 0$ on $K_1^{p^l}$. □

Corollary 1. *If K is a finite dimensional division algebra over its center then its index is equal to p.*

Corollary 2. *Suppose that a two-dimensional local skew field K splits, K_1 is a field, $K_0 \subset Z(K)$, $\operatorname{char}(K) = \operatorname{char}(K_0) = p > 2$, K is a finite dimensional division algebra over its center of index p^k.*

Then either K is a cyclic division algebra or has index p.

Proof. By [JW, Prop. 1.7] $K_1/\overline{Z(K)}$ is the compositum of a purely inseparable extension and a cyclic Galois extension. Then the canonical automorphism α has order p^l for some $l \in \mathbb{N}$. By [Zh, Lemma 10] (which is true also for $\operatorname{char}(K) = p > 0$), $K \simeq K_0((t_1))((t_2))$ with

$$t_2at_2^{-1} = \alpha(a) + \delta_i(a)t_2^i + \delta_{i+p^l}(a)t_2^{i+p^l} + \delta_{i+p^l}(a)t_2^{i+2p^l} + \dots$$

where $i \in p^l\mathbb{N}$, $a \in K_1$. Suppose that $\alpha \neq 1$ and K_1 is not a cyclic extension of $\overline{Z(K)}$. Then there exists a field $F \subset K_1$, $F \not\subset Z(K)$ such that $\alpha|_F = 1$. If $a \in F$ then for some m the element a^{p^m} belongs to a cyclic extension of the field $\overline{Z(K)}$, hence $\delta_j(a^{p^m}) = 0$ for all j. But we can apply the same arguments as in the proof of the preceding theorem to show that if $\delta_i \neq 0$ then there exists a map δ_j such that $\delta_j(a^{p^m}) \neq 0$, a contradiction. We only need to apply [Zh, Prop.2] instead of [Zh, Cor.1] and note that $\alpha\delta = x\delta\alpha$ where δ is a derivation on K_1, $x \in K_1$, $x \equiv 1 \bmod \mathcal{M}_{K_1}$, because $\alpha(t_1)/t_1 \equiv 1 \bmod \mathcal{M}_{K_1}$.

Hence $t_2 a t_2^{-1} = \alpha(a)$ and $K_1/\overline{Z(K)}$ is a cyclic extension and K is a cyclic division algebra $(K_1(t_2^{p^k})/Z(K), \alpha, t_2^{p^k})$. □

Corollary 3. *Let $F = F_0((t_1))((t_2))$ be a two-dimensional local field, where F_0 is an algebraically closed field. Let A be a division algebra over F.*

Then $A \simeq B \otimes C$, where B is a cyclic division algebra of index prime to p and C is either cyclic (as in Corollary 2) or C is a local skew field from the theorem of index p.

Proof. Note that F is a C_2-field. Then A_1 is a field, A_1/F_1 is the compositum of a purely inseparable extension and a cyclic Galois extension, and $A_1 = F_0((u))$ for some $u \in A_1$. Hence A splits. So, A is a splitting two-dimensional local skew field.

It is easy to see that the index of A is $|\overline{A} : \overline{F}| = p^q m$, $(m,p) = 1$. Consider subalgebras $B = C_A(F_1)$, $C = C_A(F_2)$ where $F_1 = F(u^{p^q})$, $F_2 = F(u^m)$. Then by [M, Th.1] $A \simeq B \otimes C$.

The rest of the proof is clear. □

Now one can easily deduce that

Corollary 4. *The following conjecture: the exponent of A is equal to its index for any division algebra A over a C_2-field F (see for example [PY, 3.4.5.])*
has the positive answer for $F = F_0((t_1))((t_2))$.

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