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REDUCTION OF MORSE FUNCTIONS ON SURFACES TO CANONICAL FORM BY SMOOTH DEFORMATION

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Relatively recently in works [3], [4] the topological classification of smooth Hamiltonian systems with one degree of freedom was obtained. When we study the stability of obtained topological invariants, the following natural question arised: is the space of all Morse functions with fixed number of minima and maxima on a closed surface connected? The present paper discusses this question and gives an algorithm of reduction of any Morse function on a closed orientable surface to the so-called canonical form.

1. Introduction

Relatively recently the works [3], [4] present the topological classification of smooth Hamiltonian systems with one degree of freedom, i. e. systems on two-dimensional surfaces. The main objects of this classification were systems on surfaces with boundary, so-called 2-atoms, see [3], [4]. This classification was received as a result of discovery of a full set of topological invariants. At once the question on stability of these invariants arised: are these invariants preserved under small perturbations of Hamiltonian systems on 2-atoms? The author in the work [8] discovered the conditions for stability of some of these topological invariants. This work lead to one more natural problem: is the space of all Morse functions with fixed number of minima and maxima on a closed two-dimensional surface connected?

Definition 1. We call Morse function *simple*, if any connected component of each level curve of this function contains at most one critical point.

It is well known fact that any Morse function on a smooth manifold can be approximated by such Morse function, which contains on each level surface at most one critical point. In other words, using arbitrary small deformation of the function, we can disrtibute all its critical points to the different levels. In this sense, any complicated Morse function can be approximated by a simple one.

It is important to note that in generally this aproximation is not uniquely defined. In other words, under different perturbations the same complicated Morse function can become a different simple Morse functions with different Reeb graphs. The Reeb graph of a function is defined as follows.

Definition 2. For any smooth function f which is defined on a surface M, let us consider the fibration (more exactly, foliation) $\pi\colon M \to W$ of this surface by connected components of level curves of the function f. The base $W = W(f)$ of this fibration we call *Reeb graph* (or *molecule*) of function f. Thus, Reeb graph shows the space of connected components of level curves of the given function. On this graph the function $f \circ \pi^{-1}$ is well-determined and, as consequence, we have a natural orientation, showing the direction of growth of this function.

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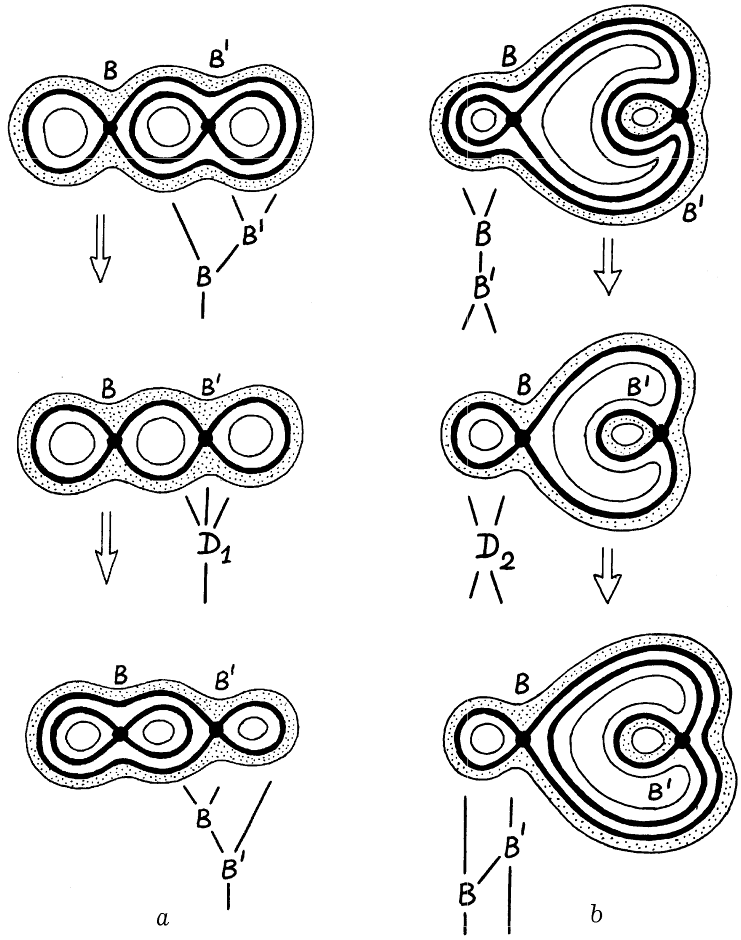

Fig. 1

Let us now discuss the following general question. Given two Morse functions f and h on the same two-dimensional closed surface, let us try to deform one to another by smooth isotopy in the space of Morse functions. The question is: when is it possible? Under which conditions on functions we can deform one of these function to another one using described deformation? One of the evident necessary conditions is as follows: functions f and h must have the same number of local minima and the same number of local maxima. Consequently, the number r of saddle critical points for these functions must be the same too. Indeed, it is clear, that $r = p + q - \chi(M)$, where $\chi(M) = 2 - 2g$ or $2 - \mu$ is Euler characteristics of the surface M; g or μ is the genus of the surface M (the number of handles in the orientable case, or the number of Mobius bands in the non-orientable case).

The necessity of the condition stated above follows from the fact that critical points do not appear or desappear in the process of an isotopy, since any "birth" or "death" of critical points is equivalent to the passing through a singularity, which is not of Morse type.

Thus, the question must be stated as follows: is the space of Morse functions with the fixed number of local minima and maxima on a given two-dimensional closed surface linearly connected? At once the natural idea arises: we shall study deformations of Morse functions by means of their molecules. In this connection, let us "represent" both of given Morse functions f and h by their molecules $W(f)$ and $W(h)$. It is not difficult to see that we can deform the first molecule to another one by a sequence of elementary transformations. At first, we can transform these molecules into simple ones (i. e. corresponding to simple Morse functions). Further, we should consider elementary transformations, transposing levels of two neighbouring vertices of the molecule, corresponding to the saddles. In the case of oriented surface, the number of all such transformations is equal to four.

These transformations can be described as shown on Fig. 1 (a,b,c,d) and 2. The Fig. 1 present the corresponding transformations of level curves of Morse functions. Thus, elementary transformations can simplify molecules of Morse functions.

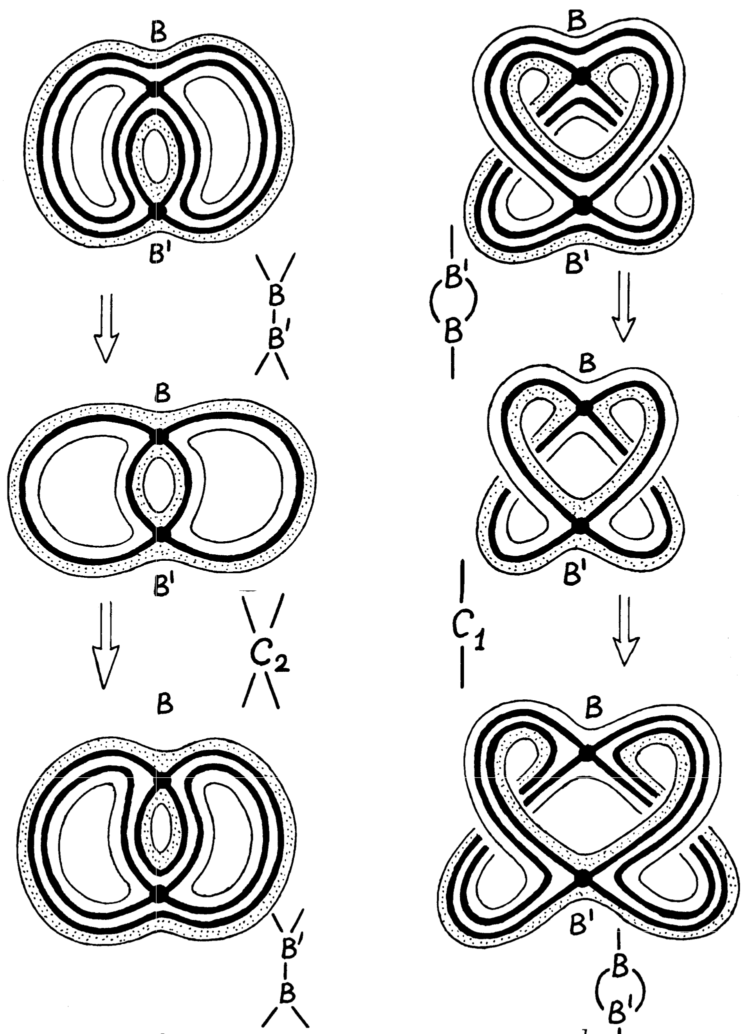

Fig. 1

Theorem 1. *Let f be any simple Morse function on two-dimensional closed orientable surface, and $W(f)$ be its Reeb graph. Then this Reeb graph always can be reduced to canonical form, which is shown on Fig.* 3, *by means of four elementary transformations described above.*

Let us note the following important property of Reeb graphs of simple Morse functions on orientable surfaces. Reeb graph $W = W(f)$ determines the simple Morse function f up to a "fiber-wise" diffeomorphism of the surface onto itself, preserving orientation. Here we call diffeomorphism fiber-wise, if it moves connected components of level curves into connected components of level curves, preserving the direction of growth of functions. Consequently, Theorem 1 implies the following statement.

Corollary 1. *Let M be a closed two-dimensional orientable surface. Then any two Morse functions f and h on M, not necessarily simple, with the same number of minima and maxima can be smoothly deformed one to another, up to some diffeomorphism of this surface, preserving orentation. In other words, there always exists such a diffeomorphism $\phi\colon M \to M$ preserving orientation and* *a smooth deformation* $f_t\colon M \to \mathbb{R}$, $0 \leqslant t \leqslant 1$, *in the space of Morse functions, such that* $f_0 = f$, $h = f_1 \circ \phi$.

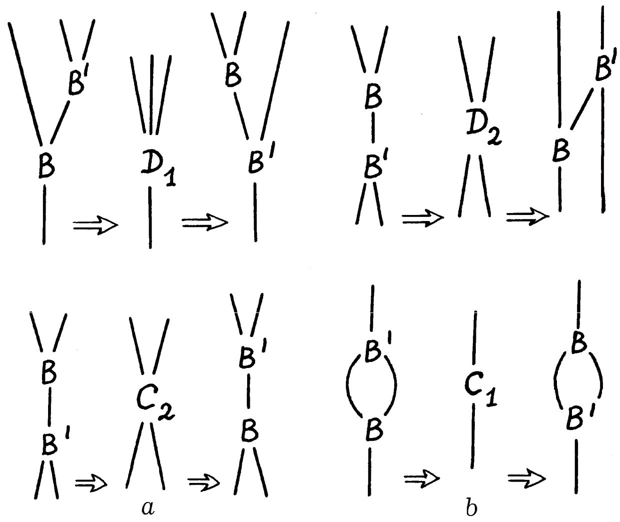

Fig. 2

Actually, this statement can be extended as follows. The space of Morse functions with fixed number of minima and fixed number of maxima on the closed two-dimensional surface is linearly connected. We did not succeed to find a proof of this statement in literature. The topological proof, known to us, was just recently obtained by S. V. Matveev. We must point out, however, that this proof is nontrivial and uses quite deep technics of low-dimensional topology. The other algebraic proof we was discovered by H. Zieschang. That proof is based on two Nielsen's theorems for free groups (see [1]) which are also nontrivial.

2. Some generalizations

Let us consider now the arbitrary closed two-dimensional surface M (orientable or non-orientable). Let us denote by $F(M,p,q)$ the space of all Morse functions on this surface, that have the fixed number p of local minima and the fixed number q of local maxima. S. V. Matveev and H. Zieschang have suggested quite different proofs of the following theorem.

Theorem 2 (S. V. Matveev–H. Zieschang). *The space* $F(M,p,q)$ *of Morse functions is linearly connected.*

REMARK 1. This theorem implies that any two Morse functions, which have the same points of minima and maxima on the surface, can be connected by some isotopy. Actually, the proofs of Matveev and Zieschang show, that this isotopy can be chosen in such a way, that all points of minima and maxima will be fixed on the surface during this isotopy.

It is useful to reformulate this theorem in terms of surfaces with boundary. Let P be a surface, where components of the boundary are separated into two classes $\partial_- P$ and $\partial_+ P$, called "negative" and "positive" circles. Let us denoted by p and q the numbers of negative and positive circles respectively.

Let $F(P)$ be the space of all Morse functions f on the surface P that have the following properties:

a) The function f has only saddle critical points on the surface P.

b) The function f has no critical points on the boundary of the surface.

c) The function f takes value $+1$ on all q positive components of the boundary and takes value -1 on all p negative components of the boundary.

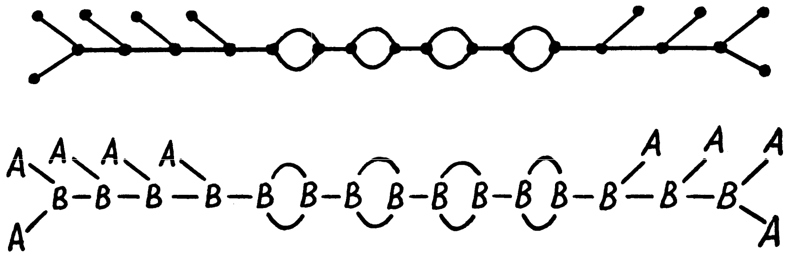

Fig. 3

Theorem 3 (S. V. Matveev–H. Zieschang). *The space $F(P)$ is linearly connected.*

Comment. In other words, if two Morse functions f_0 and f_1 have only saddle critical points on the same surface P, and, besides, take the same values on boundary components (more exactly, $+1$ on ∂_+P and -1 on ∂_-P), then we can connect this functions by a smooth path $f_t, 0 \leqslant t \leqslant 1$, in the space of similar Morse functions, i. e. in the space $F(P)$. In particular, during the isotopy, any "birth" and "death" of critical points do not occur.

Developing ideas of S. V. Matveev, the author has proved the following extention of this result. In the process of the deformation of Morse function, it is useful sometimes to control the behaviour of each critical point. In other words, sometimes we shall take into account the order on the set of critical points of function. Also, in some cases, it is useful to take into consideration the following fact. Firstly, let us fix on the surface M an arbitrary Riemannian metric. Then, for every saddle point of the function $f \in F(M,p,q)$, let us consider the smooth arc, composed by two separatrices incoming into this point. We call this curve separatrix arc. During the deformation, this arc is transformed and "interacts" with other analogous arcs. Let us fix some orientation on the arc. We can construct a deformation of the first Morse function to the second one, in a such way, that the orientations of all separatrix arcs of these functions will coincide. To answer these questions, let us introduce the following spaces of Morse functions with some "marks".

Let us consider the space $\widetilde{F}(M,p,q)$ of all Morse functions f with the following properties on the closed two-dimensional surface M.

1) The function f has p points of local minima and q points of local maxima.

2) All points of local minima and maxima of the function f are assumed to be fixed points on the surface M. More exactly, let all of these points be the same for all functions f from the space $\widetilde{F}(M,p,q)$.

3) The set of all saddle critical points of function f is enumerated, i. e. there is some fixed order on the set of all saddles of the function f.

Such Morse function f we call *function with enumerated saddles.*

It is clear, that the space $\widetilde{F}(M,p,q)$ covers the space $F(M,p,q)$. In fact, the permutation group S_r acts evidently on the described space $\widetilde{F}(M,p,q)$. Here r is the number of saddle critical points of the function f. Taking the quotient-space of the space $\widetilde{F}(M,p,q)$ under the action of this group, we obtain the space $F(M,p,q)$. In particular, the fiber of this covering is isomorphic to the group S_r. In addition, the arbitrary isotopy $f_t, 0 \leqslant t \leqslant 1$, of Morse functions in the space $F(M,p,q)$ induces an isotopy in the space $\widetilde{F}(M,p,q)$ of functions with enumerated saddles. Here we assume, that during an isotopy in the last space all critical points are moved continuously on the surface together with their enumeration.

Let us consider another space $F_+(M,p,q)$ of Morse functions f with properties 1), 2) and 4) on the closed surface M, where the property 4) is defined as follows.

4) For any saddle point the function $f \in F_+(M,p,q)$ has some fixed orientation of its separatrix arc.

We call such orientation in the saddle point *framing* of this point, and the function f itself we call

Morse function with framed saddles (saddle critical points). The obtained space of Morse functions with framed saddles we denote by $F_+(M, p, q)$. It is clear, that this space covers the space $F(M, p, q)$ with the fiber, isomorphic to the group $(\mathbb{Z}_2)^r$.

Finally, let us consider one more space $\widetilde{F}_+(M, p, q)$ of Morse functions $f \in \widetilde{F}(M, p, q)$ with enumerated and also framed saddles. In other words, each function of this space has enumeration of its saddle points, and in any saddle point it has the framing, i. e. some orientation of the separatrix arc. It is clear that the space $\widetilde{F}_+(M, p, q)$ covers the space $F(M, p, q)$ with fiber isomorphic to the group $S_r \times (\mathbb{Z}_2)^r$.

Theorem 4. *Let M be any closed two-dimentional surface. Then:*

a) the space $\widetilde{F}(M, p, q)$ of Morse functions with enumerated saddles is linearly connected;

b) the space $F_+(M, p, q)$ of Morse functions with framed saddles is linearly connected;

c) the space $\widetilde{F}_+(M, p, q)$ of Morse functions with framed and enumerated saddles is decomposed into exactly two linearly connected components.

REMARK 2. It is clear, that for any two Morse functions f and h from the same space $F(M, p, q)$, their critical points can made coincident on the surface, i. e. not only minima and maxima, but also saddles. According to Theorem 3, all points of minima and maxima can be considered as fixed on the surface during the deformation of one function to another. It is interesting to clarify: is it possible to consider all saddle points as fixed during the deformation?

3. Proof of Theorem 1

In the present paper we prove Theorem 1 about the reduction of Reeb graph to the canonical form. The proofs of the other theorems are given in the author's work [9].

Proof of the Theorem 1.

Let $M = M_g$ be a closed orientable surface of genus g. Let f_0 be a simple Morse function on this surface, see Definition 1.

For any smooth function f on M let us consider its Reeb graph $W = W(f)$ (i. e. molecule), see Definition 2. It is clear, that if f is simple Morse function, then multiplicities of all vertices of Reeb graph W are equal to 1 or 3. Besides, for any vertice of multiplicity 3, there are at least one incoming edge and also one outgoing one. Let us note, that such vertices correspond to singular level curves, containing saddle points of the function f. Evidently, all endpoints of Reeb graph correspond to points of local minima and maxima of the function f. One can prove the following realization theorem. Any continuous oriented graph W wich described form, which does not contain oriented cycles, is Reeb graph of a suitable simple Morse function on the closed orientable surface M. In addition, the genus g of the surface M coincides with the genus of the initial graph W.

Let p be the number of local minima and q be the the number of local maxima of the given function f_0. Let us denote by $W(g, p, q)$ the oriented graph having p "lower" endpoints and q "upper" endpoints, such that this graph is composed by g circles, which are connected by segments one by one, and also two "trees" are added, see Fig. 3. This graph is uniquely determined by numbers p and q.

Let us assume, that the function f_0 has at least two local minima, i. e. $p > 1$.

Lemma 1. *Let us fix any two lower endpoins of Reeb graph W. Then there exists a finite sequence of admissible transfomations of this graph, see Fig.* 2, *such that the given two points can be connected by path consisting of two neighbouring edges in the obtained graph.*

Proof.

Let us consider the arbitrary path $\gamma \subset W$ connecting the given endpoints. Without loss of generality, we can assume that γ is simple path, i. e. it is not self-intersecting. Let us show, that untill the lenght of this path is two, this lenght can be decreased by one with the help of admissible transformations. In fact, since the both endpoints of the initial path γ are lower endpoints of W, its interior always has at least one "maximal" point. It is not difficult to see that some neighborhood of this point in γ has the form representing on Fig. 4 (a,b,c). These figures show also how one can decrease by one the lenght of the path γ, using transformations D_1, D_2, demonstrated on Fig. 2. Lemma is proved. ∎

Now let us return to the proof of the Theorem 1. Let $N = 2(p+q+g-1)$ be the number of vertices of the given graph W. Let us prove the theorem by induction on N. The base of induction is evident, since under $N = 2$ or $N = 4$ all functions $f \in F(M_g, p, q)$ have the same Reeb graph $W(g,p,q)$. In order to realize inductive step from $N-2$ to N, let us consider three cases: (a) $p > 1$, (b) $q > 1$ and (c) $p = q = 1$.

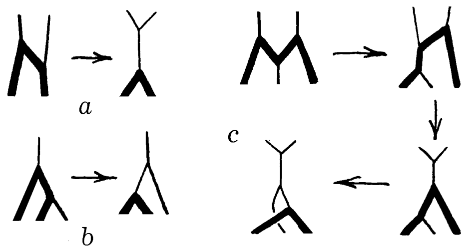

Fig. 4

In the case (a) we conclude from lemma, that any two lower endpoints of the graph W can become endpoints of two neighboring eges, after a suitable sequence of admissible transformations of this graph. Removing this two edges from the graph, we obtain new Reeb graph having $N-2$ vertices. By inductive hypothesis, this graph can be reduced to the canonical form $W(g, p-1, q)$. Consequently, restoring two removed edges, we obtain the reduction of W to the canonical form $W(g,p,q)$.

The case (b) is considered in analogous way.

Finally, let us consider the case (c). Since $N \geqslant 4$ and $p = q = 1$, we have $g \geqslant 1$. Let us remove from Reeb graph the unique lower endpoint both with edge outgoing from it. As a result, we obtain new Reeb graph of genus $g-1$, which has the same number N of vertices, but two of these vertices are lower endpoints. Consequently, by item (a), proved above, the obtained graph can be reduced to the canonical form $W(g-1,2,1)$. Restoring the removed edge, we obtain the requied reduction of W to the canonical form $W(g,1,1)$.

Theorem 1 is proved. ∎

REMARK 3. A simple modification of this proof shows, that the following proposition about Reeb graphs with enumerated vertices is valid. Let us consider a simple Morse function f with enumerated vertices. In other words, we assume that on each of three sets of critical points: minima, maxima and saddles, there is some fixed order, i.e. enumeration. It is clear, that under the projection $\pi\colon M \to W$ of the surface onto Reeb graph, this enumaration is moved into some enumeration of vertices of Reeb graph $W = W(f)$. Let us denote by $\widetilde{W}$ this graph with induced enumeration of its vertices. We can show, that Reeb graph $\widetilde{W}$ can be reduced to the canonical form, see Fig. 3. Here vertices of the canonical Reeb graph are enumerated by arbitrary way. In fact, the proof of the Theorem 1 shows, that enumeration of all endpoints of Reeb graph can be reduced to the canonical form in $W(g,p,q)$. Thus, actually we needto prove, that any permutation of interior vertices of the canonical graph can be realized by some suitable sequence of admissible transformations, under which all endpoints are "fixed". Since canonical Reeb graph is connected, we can restrict ourself only to transpositions of neighbouring interior vertices. Any such transposition can be easily realized by transformations C_1, C_2 and D_1, see Fig. 2.

In conclusion, let us recall a corollary from the Theorem 1. The proof of this theorem gives an algorithm, which allows to construct an isotopy connecting arbitrary simple Morse function f with Morse function of the "canonical" form on the orientable closed surface. Here we say, that simple Morse function has canonical form, if its Reeb graph coincides with $W(g,p,q)$. This algorithm always stops after a finite number of steps.

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