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GENERALIZATION OF GEOMETRIC POINCARE THEOREM FOR SMALL PERTURBATIONS.

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We consider the dynamical system, which phase space containes a closed submanifold filled by periodic orbits. The following problem is analysed. Let us consider a small perturbations of the system. What we can say about the number of periodic orbits, survived under perturbation, and about their location in the neighborhood of the submanifold under consideration? We obtain the solution of this problem for the perturbations of general type in terms of avereged perturbation. The main result of the paper is as follows. Theorem: Let us consider the Hamiltonian system with Hamiltonian function H on a symplectic manifold (M^{2n}, ω^2). Let $\Lambda \subset H^{-1}(h)$ be the closed nondegenerate submanifold filled by periodic orbits of this system. Then for the arbitrary perturbed function $\tilde{H}$, which is C^2–close to the initial function H, the system with the Hamiltonian $\tilde{H}$ has at least two periodic orbits on the isoenergy surface $\tilde{H}^{-1}(h)$. Moreover, if either the fibration of Λ by closed orbits is trivial, or the base $B = \Lambda/S^1$ of this fibration is locally flat, then the number of such orbits is not less than the minimal number of critical points of smooth function on the quotient manifold B.

1. Introduction

This papers contains two main results. The first one can be considered as the new and more simple proof of partial results of A. Weinstein [16]. The second one is the generalization of the main result of the paper [8] obtained by J. Moser, and it was obtained by Weinstein in the paper [9].

Our results also confirm the well-known V. I. Arnold conjecture that the geometric H. Poincaré theorem [1], [12], [27] can be generalized on the case of arbitrary symplectic manifolds and arbitrary perturbations. Let us note that our result do not give yet the complete generalization of Poincaré theorem for the arbitrary perturbations, because we have proved this general result only for small perturbations. It is possible that this result can be extended with the help of technique of papers [8], [16]–[26].

Besides, the first our result (theorems 1, 3) generalizes the result of [24] by P. L. Ginzburg. The difference between our work (see also works [9], [10], [15], [16] by Weinstein) and the paper [24] is as follows. Ginzburg in [24] had considered the periodic systems and obtained the estimation for the number of periodic orbits for perturbed systems. It turns out (see [9]) that the same estimation take place also in general case. Namely, when the system has submanifold filled by the periodic solutions of the unperturbed system.

The second our result (theorems 2, 4) is formulated as the averaging principle on a submanifold [9]. This result is the generalization of the averaging principle on a manifold, obtained by Moser in the paper [8]. In this paper Moser had considered periodic systems.

Further, we formulate two statements (propositions 2, 3) generalizing theorem 1 in the following cases. 1) When the submanifold, filled by periodic orbits, contains equilibrium points of the system. 2) When the symplectic structure is perturbed analogously to Hamiltonian. In the first of this statements we consider the situation which partial cases had been studied by Weinstein [9], [10] and Moser [13]. In these works the submanifold considered by us coincides with an equilibrium point, and some spetial estimation is proved for the number of periodic orbits near an equilibrium. The second our generalization is illustrated in separate chapter, when we discuss some author's result on celestial mechanics. In this situation the initial symplectic structure is degenerate, while the generalised Poincaré theorem remains valid under admissible perturbations of the symplectic structure.

Then, we give the formulations of both our theorems 1 and 2 (i. e. generalized geometric Poincaré theorem and the averaging principle on the submanifold) for dynamical systems of general type, i. e. not necessarily Hamiltonian ones. The close results have been obtained also by F. B. Fuller [6] and Moser [8].

Both results of the present paper are applied not only for the case of standard, regular fibrations, but also for the fibrations with singular fibers (i.e. singular fibrations). Namely, for the multidimensional Seifert fibrations. Such fibrations appear, for example, in the analysis of periodic solutions far from the equilibrium points of the system (see works [9], [10], [13] and [15]). In our paper we introduce the notion of the circular functions on Seifert fibration. We give an estimation for the number of periodic solutions of the perturbed system in terms of such functions.

Let us note that for arbitrary fibrations our estimation seems to be weaker than Weinstein's one [16], because circular functions not always can be projected onto the base of Seifert fibration. In addition, in the paper [16] Weinstein had considered more wide class of perturbations. Namely, he assumed that C^1-norm of the perturbation is small. In addition to this assumption, we assume that C^2-norm of the perturbation is also small. But our proof is more simple and more geometric. Developing Poincaré's ideas, we are based on the classical implicit function theorem. In particular, in our present paper we do not consider the infinite-dimensional space of loops, in contrast with the paper [16]. Let us note, that, by our technique, for the wide class of fibrations we succeeded in proving the Weinstein's estimation. Namely, this is the class of fibrations, which bases are locally flat, for example, tori.

2. Generalized geometric Poincaré theorem

In this section we formulate the main result of the paper. In the following section we will formulate the averaging principle and give proofs. Let us consider the Hamiltonian system with Hamiltonian function H on symplectic manifold (M^{2n}, ω^2). Let $\Lambda \subset H^{-1}(h)$ be a closed submanifold filled by periodic orbits of this system. We assume that the surface $H^{-1}(h)$ is regular, i.e. without singular points. Let us consider the orientable smooth fibration

$$S^1 \to \Lambda \xrightarrow{p} B \tag{1}$$

with fiber circle. All these fibers are assumed to be periodic orbits on Λ, and the base B is the quotient manifold with natural quotient topology. We'll assume that Λ is not an isolated periodic orbit, i. e. its dimension is greater than 1.

Let us assume that there is the continuous (and consequently smooth) function on Λ, which is the period function $T : \Lambda \to \mathbb{R}$ for the periodic orbits on this submanifold. This means that on Λ there is the smooth action of the circle. This action foliates Λ into the regular and singular fibers, which are homeomorphic to the circle. The orbits if this action coincide with periodic orbits of the given system on Λ. The time of motion along these orbits is prorortional to the natural parameter on the circle with the factor $\frac{T}{2\pi}$. In particular, this action is locally free (i. e. has no fixed points), because we have assumed the regularity of the isoenergy surface.

Such fibration of Λ by circles we call *periodic fibration*, or *Seifert fibration*. The function T is not necessarily minimal period for some orbits, which are called *singular fibers*. In this case the function T has the form $T =$ (minimal period) $\cdot k$, where k is some integer number depending on the fiber. In the case of locally trivial fibration the function T is equal to the minimal period everywhere on Λ, and the quotient manifold $B = \Lambda/S^1$ is smooth. Let the periodic fibration (1) has singular fibers. Then the quotient B will no longer be smooth manifold, but it has the structure of so called *generalised manifold*, or *V–manifold* [4], [5], [15]. Namely, the function f on B is called smooth if the inverse image $p^*f : \Lambda \to \mathbb{R}$ of this function is smooth function on Λ. Thus, critical points of the tunction f coincide with projections of critical orbits for the function p^*f. We call critical point of function f nondegenerate, or Morse point, if the corresponding critical orbit of function p^*f is Bott critical circle. Let us note that there is natural well-defined closed 2–form on the considered V–manifold. Namely, it is the "projection" on B of the restriction of the symplectic structure to the submanifold Λ. This 2–form can also be a symplectic dtructure on B, see below.

Let us consider for arbitrary orbit $\gamma \subset \Lambda$ the small transversal cross-section $\sigma \subset M_h$ called *Poincaré section*. Then we consider the Poincaré mapping $A : \sigma \to \sigma$ from this cross-section into itself. This mapping is determined by the flow corresponding to the initial system. The time of the motion along the orbit from the initial point on the cross-section to the image of this point under Poincaré mapping is approximately equal to the period $T|_\gamma$. It is clear that the intersection point $m = \gamma \cap \sigma$ of initial orbit γ with the cross-section σ is a fixed point of the Poincaré mapping A. Let us consider the linearization $dA(m)$ of this mapping in the fixed point m.

Definition 1. The submanifold Λ is called *nondegenerate* if the fibration (1) is periodic, and for any orbit $\gamma \subset \Lambda$ the kernel of the operator $dA(m) - I$ coincides with the tangent space $T_m(\Lambda \cap \sigma)$ to the submanifold $\Lambda \cap \sigma$:

$$\ker(dA(m) - I) = T_m(\Lambda \cap \sigma),$$

where I is the identity operator in the tangent space $T_m\sigma$ to the Poincaré section σ.

Let us consider on (M^{2n}, ω^2) the perturbed system with the Hamiltonian $\tilde{H}$, which is close to H with respect to the norm C^2.

Definition 2. In the present paper, speaking about a *closed orbit* of the perturbed system, we always assume that this closed orbit satisfies an additional condition of "almost T–periodicity". This means that this orbit is close to the submanifold Λ, and its period is close to the period $T|_\gamma$ of some unperturbed orbit $\gamma \subset \Lambda$, which is close to the orbit $\tilde{\gamma}$.

We call the manifold (or generalized manifold) *locally flat*, if this manifold has locally flat affine connectedness. We consider at first the case of periodic fibrations of special type. Namely, the case of trivial fibration and the case of fibration with locally flat base.

Theorem 1. *Let the submanifold $\Lambda \subset M_h$ be filled by the closed orbits of the unperturbed system. Let this submanifold be closed (i. e. compact and without boundary), and is nondegen-rate. Let the fibration (1) of the submanifold Λ by periodic orbits be periodic and has special type, namely: A) either this fibration is trivial, B) or the base $B = \Lambda/S^1$ of this fibration is locally flat (for example, is diffeomorphic to the multidimensional torus).*

Then the number of geometrically different closed orbits of perturbed system on the isoenergy surface $\tilde{H}^{-1}(h)$ is not less than the minimal number of critical points of the smooth function on the quotient manifold B. Besides, the number of such orbits with their multiplicities is not less than the minimal number of critical points of Morse functions on B.

As the corollary, we obtain, that if the fibration p is of special type (either trivial fibration or the fibration with locally flat base), then the number of geometrically different orbits of the perturbed system is not less than the Lusternik-Schnirelman category $\operatorname{cat} B$ of the quotient manifold B. Becides, if such fibration is locally trivial, then the number of such orbits is not less than the sum $\sum \beta_i(B)$ of Betti numbers of the manifold B.

REMARK 1. *Locally flat manifolds, different from the tori, really exist, and their number is sufficiently large, see Wolf's book [29]. Let us remark also that the periodic fibration over locally flat manifold is not necessarily trivial. The restrictions on the topology of the fibration (1) are indeed not so important, i. e. the theorem 1 is valid also for arbitrary periodic fibrations, see [16].*

Now we consider in more details the case when Seifert fibration p is not necessarily trivial. We introduce the notion of circular functions on the manifold Λ. Let us denote by v the vertical vector field on Λ, tangent to fibers of the fibration p, i. e. v is the restriction of the given unperturbed system on the submanifold Λ. Let $\tilde{v}$ be some vector field on Λ, which is close to the vertical field v. Speaking about the nearness of vector fields, we always mean their nearness with respect to the C^0–norm. The definition of almost T–periodic orbits of the field $\tilde{v}$ is analogous to the definition 2. Let F be a smooth function on Λ, which is the first integral of the field $\tilde{v}$, i. e. the function F is constant along each orbit of this field.

Definition 3. Such function F on Λ is called *circular function* for the field $\tilde{v}$, if any orbit of this field, which is critical for F, is closed orbit. In other words, this orbit is closed and almost T–periodic orbit of the field $\tilde{v}$. We call such an orbit *critical circle* of the circular function F. If, in addition, all critical circles of the function F are Bott critical submanifolds for this function, then we call the function F Bott circular function.

Further, we will call a function F on Λ circular, if there exists some vector field $\tilde{v}$, close to vertical field v, and satisfying to all properties listed above. In other words: 1) the function F is constant along all orbits of the field $\tilde{v}$, and 2) the function F if the circular function for this field.

Let us note some properties of circular functions on the space Λ of Seifert fibration. a) The class of circular functions includes the class of all smooth functions on the quotient manifold B, because any function on B is "the image" of some circular function on Λ for vertical field v. b) Critical submanifolds of arbitrary circular function with finite number of critical circles, are smooth closed curves (circles) which are close to the fibers of the fibration p. c) It is easy to see, that for any nondegenerate singular fiber γ of Seifert fibration there exists the close orbit $\tilde{\gamma}$ of the field $\tilde{v}$. And such a orbit is critical for circular function F. Here we call the fiber γ of Seifert fibration nondegenerate, if the following conditions are valid. Let us consider the small cross-section in Λ for γ, and consider the Poincaré mapping of this cross-section onto itself, which is determined by the flow of field v on Λ. If the spectrum of the differential of Poincaré mapping does not contain 1, then such an orbit is called nondegenerate for the fibration (1). In analogous way, we can define the notion of nondegenerate submanifold in Λ, filled by singular fibers.

Theorem 1*. *Let $\Lambda \subset M_h$ be the submanifold, filled by closed orbits of the system with Hamiltonian H. Let this submanifold be nondegenerate, closed (i. e. compact, without boundary) submanifold as in the theorem 1. But, in contrast with the theorem 1, the periodic fibration (1) of this submanifold is not necessarily trivial. Then: A) There are at least two closed orbits of the perturbed system on the surface $\tilde{H}^{-1}(h)$. B) Moreover, the number of such orbits is not less that the minimal number of the critical circles of the circular function on the manifold Λ. Also, the number of such orbits, counted with their multiplicities, is not less than the minimal number of critical circles of Bott circular function on Λ.*

As the corollary, we obtain the following rough estimation for the number of closed orbits of the perturbed system on the surface $\tilde{H}^{-1}(h)$. Namely, the number of geometrically different closed orbits of perturbed system on the surface $\tilde{H}^{-1}(h)$ is not less than $\frac{1}{2}\operatorname{cat}\Lambda$. If the fibers of the fibration p can be contracted into the point, then the number of such orbits is not less than $\operatorname{cat}\Lambda$. If the fibration p is locally trivial, then the number of such orbits, counted with their multiplicities, is not less than $\frac{1}{2}\sum\beta_i(\Lambda)$. Let us note, that the estimation in terms of the base B, analogous to the estimation obtained from the theorem 1, does not follows from this result, because the following inequalities are valid: $\frac{1}{2}\sum\beta_i(\Lambda) \leq \sum\beta_i(B)$, $\dfrac{\operatorname{cat}\Lambda}{cat_\Lambda S^1} \leq \operatorname{cat} B$, where S^1 is the fiber of the fibration (1). There are fibrations, for which these inequalities become strong.

Actually, as it was mentioned above, the estimation from the theorem 1 is valid for arbitrary fibrations, because the estimation in terms of critical points of smooth functions on the base B is valid in general case. See [14]. Let us note that the item A) of the theorem 1 is the simple corollary of the theorem 1*.

Let us underline that we do not restrict the structure of the quotient manifold B with natural 2–form on it, and do not restrict its dimension and its topology, except of the case B) of theorem 1. In particular, our manifold B is not necessarily symplectic, not necessarily isotropic, and not necessarily orientable. But in some special cases the theorem 1* gives us more convenient and more concrete estimation. For example, Moser [13] and Weinstein [15] noted that $\operatorname{cat} B \geq \frac{1}{2}\dim B + 1 = \frac{1}{2}(\dim\Lambda + 1)$ in case when either the quotient manifold B is symplectic, or (which is the same) the multiplicity of 1 in the spectrum of the operator $dA(m)$ equals $\dim B$. More surprising and interesting fact is that, the previous estimation is valid also in the case when Λ is homotopically equivalent to the odd-dimensional sphere. See the papers of Krasnosel'skii [3] and Weinstein [10]. The second of these papers proves the corresponding estimation for the number of closed orbits of perturbed system, without the complicated technique used in the paper [16]. Here we call the quotient manifold B symplectic with respect to the natural 2–form on it, if the restriction of the initial symplectic structure ω^2 to the intersection of the cross-section σ and Λ is nondegenerate 2–form.

REMARK 2. *Let the fibration (1) be not locally trivial, i. e. it has singular fibers. It turns out that many closed orbits of perturbed system has minimal period close to the number T/k, but not to T, where k is an integer number greater than 1. Indeed, let us consider all orbits on Λ, which periods are factors of T/k. Let us assume that the periodic fibration is nondegenerate, i. e. for any k each connected component of union Λ_k of all such singular fibers is smooth submanifold in Λ. Besides, let this union be nondegenerate in the fibration (1) with respect to natural sense. Then periods of corresponding closed orbits of perturbed system be close to T/k. Indeed, we can apply theorem 1 to each of this components with period function T/k on Λ_k. Thus, for nondegenerate periodic fibrations minimal periods of many closed orbits of perturbed system will be close to T/k, but not to T, where $k > 1$. In particular, each isolated nondegenerate fiber remains under perturbation. In other words, this orbit generates the closed orbit of perturbed system with period closed to minimal period of the unperturbed orbit.*

3. The averaging method on a submanifold. Proof of main results

Let us assume that, under the hypothesis of theorem 1*, the initial Hamiltonian system is included into one-parameter family of perturbed Hamiltonian systems depending on small parameter ϵ with Hamiltonian

$$\tilde{H} = H + \epsilon H_1 + o(\epsilon), \qquad \epsilon \to 0. \tag{2}$$

It turns out [9], that in the case of general perturbation under small $|\epsilon|$ we can estimate the number of closed orbits of perturbed system on the surface $\tilde{H}^{-1}(h)$ more exactly. Namely, we'll "average" the perturbation H_1 over the periodic orbits on Λ and look for critical submanifolds of the obtained function on Λ. These critical orbits generate closed orbits of perturbed system. This is essentially the content of the averanging method on a submanifold which was given in this formulation by Weinstein in [9]. In the present paper we give the proof different from Weinstein's one. Our technique goes back to Poincaré [1]. Also it has been exploited by G. Reeb [2], Moser [8] and G. A. Krasinskii [11]. Let us note that this method generalises the averaging method in Euclidean space [2], on a manifold [8] and on Liouville tori [1], [11].

We pass now to more exact formulations. Let us consider the restriction $H = H_1|_\Lambda$ of the perturbation to Λ. The following function:

$$\bar{H}(m) = \int_0^{T(m)} H(\gamma(m,t))dt, \qquad m \in \Lambda, \tag{3}$$

we'll call *averaged perturbation* on Λ. Here $\gamma(m,t)$ meens the periodic solution of unperturbed system with initial value $\gamma(m,0) = m$, and $T : \Lambda \to \mathbb{R}$ is the continuous function of periods on Λ. It is clear that the obtained function $\bar{H}$ on Λ is constant along each fiber of fibration (1). Consequently, there is well-defined "projection" of this function onto the quotient B. The following theorem shows that Bott critical orbits of function $\bar{H}$ give rise to closed orbits of perturbed system.

Theorem 2. *Let $\Lambda \subset M_h$ be a submanifold, filled by closed orbits of the unperturbed system. Let this submanifold be nondegenerate, but not necessarily compact. Let the perturbed Hamiltonian $\tilde{H}$ depends smoothly on a small parameter ϵ, i. e. it has the form (2). And let $\gamma_0 \subset \Lambda$ be Bott critical orbit of the averaged perturbation $\bar{H}$, see (3). Then there exists one-parameter family of closed orbits $\gamma_\epsilon \subset \tilde{H}^{-1}(h)$ of the perturbed system. This family depends smoothly on the parameter of perturbation ϵ under small ϵ, and γ_ϵ coincides with γ_0 under $\epsilon = 0$.*

In cases when the function $\bar{H}$ is not constant and not necessarily Bott, one can apply the following statements.

Statement 1-A. *Let, under the hypothesis of theorem 2, $\gamma \subset \Lambda$ be some orbit of the unperturbed system. Let this orbit be not critical for the averaged perturbation $\bar{H}$, see (3). Then there exists such a neighborhood U of this orbit in M^{2n}, that for sufficiently small ϵ on the surface $U \cap \tilde{H}^{-1}(h)$ there are no closed orbits of the perturbed system with Hamiltonian $\tilde{H}$.*

In order to formulate the second the statement, we introduce the following notion.

Definition 4. Let m be an isolated critical point of the smooth function $F : \mathbb{R}^k \to \mathbb{R}$, $F(m) = 0$. Let us consider the subcritical set $M = \{x \in \mathbb{R}^k | F(x) \leq 0\}$ of this function. The following integer numbers: $\beta_i(F,m) = rank\, H_i(M, M \backslash \{m\})$, $0 \leq i \leq k$, we call Betti numbers of the function F in the point m.

It can be shown that the index of gradient of this function in the point m equals the alternated sum $ind_m(\nabla F) = \sum(-1)^i \beta_i(F,m)$ of Betti numbers. Besides, if m is Morse critical point, then only one of Betti numbers is nonzero. More exactly, in this case $\beta_i(F,m) = 1$ if i is equal to the index of Hessian $d^2F(m)$, and $\beta_i(F,m) = 0$ for all other i.

Statement 1-B. *Let, under the hypothesis of theorem 2, $\gamma \subset \Lambda$ be (not necessarily Bott) isolated critical orbit for the function $\bar{H}$ on Λ. Let us consider the restriction $F = \bar{H}|_{\bar{\Lambda}}$ of this function onto the cross-siction $\bar{\Lambda}$ to γ in Λ. Let $\beta_i(F,m)$ be Betti numbers of the obtained function in the intersection point $m = \gamma \cap \bar{\Lambda}$ of γ with this cross-section. Let us suppose, that at least one of this numbers is nonzero, and consequently their sum $\beta = \sum \beta_i(F,m)$ is positive. Then for any preassigned neighborhood U of the orbit γ in M^{2n}, for sufficiently small ϵ, on the surface $U \cap \tilde{H}^{-1}(h)$ there is at least one closed orbit of system with Hamiltonian $\tilde{H}$ of the form (2). In addition, the number of such orbits in $U \cap \tilde{H}^{-1}(h)$, counted with their multiplicities, is at least β.*

Proof of theorems 1-A, 1, 2 and statements 1-A, 1-B.*

Now we'll formulate the last statement, which implies all statements listed here. Let us assume, that Hamiltonian $\tilde{H}$ of the pertutbed system is close to H with respect to C^r–norm, where $r \geq 2$. Let is denote $\epsilon = \|\tilde{H} - H\|_{C^r}$.

Statement 1-C. *Let $\Lambda \subset M_h$ be nondegenerate submanifold, filled by closed orbits of the unperturbed system. Then there exists such a neighborhood U of this orbit in M^{2n}, depending only on the unperturbed system, that for any sufficiently small $\epsilon > 0$ there exists an embedding $i : \Lambda \hookrightarrow U$ and a circular function S on the submanifold Λ, which have the following properties:*

1) The image $\tilde{\Lambda} = i(\Lambda)$ of the submanifold Λ under the embedding i lies on the surface $\tilde{H}^{-1}(h)$. 2) Images under the embedding i of all critical circles of function S coincide with closed orbits of the perturbed system on $U \cap \tilde{H}^{-1}(h)$. In particular, the submanifold $\tilde{\Lambda}$ containes all closed orbits of perturbed system on $U \cap \tilde{H}^{-1}(h)$. 3) The embedding i is close to identity mapping, and the function $-S$ is close to function $\bar{H}$, obtained by averaging (3) of the perturbation $H_1 = (\tilde{H} - H)/\epsilon$ over the closed orbits of the unperturbed system on Λ. 4) If the perturbed Hamiltonian $\tilde{H}$ smoothly depends on the small parameter, then both embedding $i : \Lambda \hookrightarrow M$ and function S on Λ together with the corresponding to it vector field $\tilde{v}$, close to the vertical, depend smoothly on this parameter.

Here we call two mappings or functions close to each other if they are ϵ–close with respect to C^{r-1}–metrics, and the nearness of two vector fields we consider with respect to C^{r-2}–norm.

Proof.

We split the proof of statement 3 into several steps. Step 1. Let us fix some affine connectedness on the isoenergy surface $H^{-1}(h)$. Let $\sigma_m \subset H^{-1}(h)$ be a small cross-section in the isoenergy surface to the orbit passing through the point $m \in \Lambda$. Let us consider (depending on m) Poincaré mapping A of the cross-section σ_m into itself. Further, let us construct the co-kernel of the operator $dA(m) - I$. More exactly, let us denote by $D_m = Im(dA(m) - I)$ the image of this operator. Let surface $\theta_m \subset \sigma_m$ be transversal to this the image D_m in σ_m and has the same dimention as $\Lambda \cap \sigma_m$. We'll assume that both constructed surfaces σ_m and $\theta_m \subset \sigma_m$ are composed by geodesics passing through the point m and depend smoothly on the point $m \in \Lambda$.

Now let us fix a small neigborhood U of the submanifold Λ in M and denote by U_h the intersection of this neighborhood with the isoenergy surface $H^{-1}(h)$. We'll assume that both of these surfaces σ_m and $\theta_m \subset \sigma_m$ are determined for all points $m \in U_h$, composed by geodesics and smoothly depend on the point. Let us transfer the affine connectedness and both constructed fields of surfaces onto the "perturbed" isoenergy surface $\tilde{H}^{-1}(h)$, using some diffeomorphism $H^{-1}(h) \to \tilde{H}^{-1}(h)$ close to the identity mapping. Let us denote $\tilde{U}_h = U \cap \tilde{H}^{-1}(h)$.

Step 2. Let us consider in the neigborhood U the restriction of the perturbed system onto the isoenergy surface $\tilde{H}^{-1}(h)$. From each point $m \in \tilde{U}_h = U \cap \tilde{H}^{-1}(h)$ of this surface we'll move along the integral curve $\tilde{\gamma}_m \ni m$ of the perturbed system during the time, close to the period $T|_\gamma$ of the orbit $\gamma \subset \Lambda$ close to $\tilde{\gamma}_m$. Let us denote by $\tilde{A}(m)$ the intersection point of this orbit with the cross-section $\tilde{\sigma}_m$. The obtained mapping $\tilde{A} : \tilde{U}_h \to \tilde{U}_h$ we call the perturbed Poincaré mapping. It is clear, that for the unperturbed system this mapping coincides with unperturbed Poincaré mapping $A : U_h \to U_h$. Let us note several properties of Poincaré mapping: A) Fixed points of Poincaré mapping $\tilde{A}$ coincide with points in $\tilde{U}_h$, lying on the closed orbits of the perturbed system. B) Poincaré mapping is symplectic, i. e. $\tilde{A}$ preserves the restriction of the symplectic structure onto the isoenergy surface $\tilde{H}^{-1}(h)$. C) Poincaré mapping "preserves the center of mass". More exactly, for any closed curve $\gamma \subset \tilde{U}_h$, the integral of 2–form $\omega = \omega^2$ over a chain $\Gamma, \partial\Gamma = \gamma - \tilde{A}(\gamma)$, "relating" this curve and its image $\tilde{A}(\gamma)$ under Poincaré mapping, vanishes:

$$\iint_\Gamma \omega = 0.$$

Here 2–chain Γ is homologous to the "cylinder", which is composed by the geodesic segments relating each point $m \in \gamma$ with its image $\tilde{A}(m)$ in $\tilde{H}^{-1}(h)$.

To prove the last property of Poincaré mapping (i. e. preserving of the center of mass), let us consider the differential 2–form $\tilde{\Omega} = \omega - d\tilde{H} \wedge dt$ in the extended phase space $M^{2n} \times \mathbb{R}_t$. As far as the whole 2–chain $\Gamma \subset M^{2n} \times \mathbb{R}_t$ lies intirely in the isoenergy surface $\tilde{H}^{-1}(h)$ in the extended phase space, the results of integration over this chain of forms $\tilde{\Omega}$ and ω will be equal to each other. Let us close the chain Γ by the tube composed by orbits passing through the initial curve γ. The obtained 2–cycle we denote by $[\Gamma]$. By virtue of Hamilton's principle of the least action, the restriction of 2–form $\tilde{\Omega}$ on the added tube is zero. Therefore, we actually must prove that the integral of this 2–form over the obtained 2–cycle $[\Gamma]$ vanishes. For proving this fact let us notice, that 2–form $\tilde{\Omega}$ is co-homologous to unperturbed 2–form Ω of the same type, and that homologous curves γ give rise to homologous 2–cycles $[\Gamma]$ in $M^{2n} \times \mathbb{R}_t$. From these facts we conclude that it is enough to verify the "preservation of the center of mass" only for the unperturbed system. And moreover, we may restrict ourself only to curves γ lying in Λ, because the value of the integral under consideration depends only on the homology class of the curve $\gamma \subset M^{2n}$. It remains to notice that for any such curve $\gamma \subset \Lambda$ the chain Γ is "degenerate", since it turnes into the curve γ. Thus, the integral of the symplectic structure over such a chain vanishes.

Step 3. Let us determine the perturbed submanifold $\tilde{\Lambda}$ as the set of all points $m \in \tilde{U}_h$, such that $\tilde{A}(m) \in \tilde{\theta}_m$. It is easily to see that for the unperturbed system this set coincides with Λ. By the implicit function theorem, for sufficiently small ϵ, this set has the form $\tilde{\Lambda} = i(\Lambda)$, where i is an embedding close to the identity mapping. Indeed, the implicit function theorem is applicable, because, in according to the construction of the surfaces θ_m, $m \in \Lambda$, their tangent spaces $T_m\theta_m$ are tranversal in $T_m\sigma_m$ to images of operators $dA(m) - I$, where A is the unperturbed Poincaré mapping, I is the identity operator in $T_m\sigma_m$.

Let us note that, by definion of the set $\tilde{\Lambda} \subset \tilde{U}_h$, the embedding i satisfies the property 1) from statement 1-C, and all closed orbits of the perturbed system in $\tilde{U}_h$ are automatically contained in the constructed submanifold $\tilde{\Lambda} = i(\Lambda)$.

Step 4. The following our constructions are completely analogous to Ginzburg's ones, see [16], where only periodic systems were studied. Let us construct in $\tilde{U}_h$ the smooth function Ψ by the following way. This function is determined up to an additive constant. The difference of its values in any two points $m_0, m_1 \in \tilde{U}_h$ is equal to the integral of 2–form ω over the chain $\Gamma(m_0, m_1)$, composed by geodesic segments relating each point m_t with its image $\tilde{A}(m_t)$ under Poincaré mapping, where $m_t, 0 \leq t \leq 1$, means any path in $\tilde{U}_h$ relating points m_0 and m_1:

$$\Psi(m_0) - \Psi(m_1) = \iint_{\Gamma(m_0,m_1)} \omega.$$

The last property C) of Poincaré mapping (preservation of the center of mass) implies that the smooth function Ψ is global single-valued function in $\tilde{U}_h$, although it is determined up to an additive constant. For definiteness, let us assume, that this function vanishes in the point $i(m_0)$, where m_0 is some fixed point in Λ. It is clear that the determined in such a way unperturbed function Ψ identically vanishes on Λ. Let us denote by ϵS the function $\Psi \circ i$ on the submanifold Λ, where i is the embedding constructed on the previous step.

Step 5. On this step we give an explicit representation for differential of the function S, and show that this function is circular. Let us denote by $g(m,t), 0 \leq t \leq 1$, the geodesic segment relating the point $m = g(m,0) \in \tilde{U}_h$ with its image $\tilde{A}(m) = g(m,1)$. For any curve $g(t)$, $0 \leq t \leq 1$, in $\tilde{U}_h$ we denote by $\xi_t \in T_{g(t)}\tilde{U}_h$, $0 \leq t \leq 1$, the parallel carry of the tangent vector $\xi_0 \in T_{g(0)}\tilde{U}_h$ along this curve. It follows from the definition that the differential of the function Ψ in an arbitrary point $m \in \tilde{U}_h$ is equal to

$$d\Psi(m)\eta = \int_0^1 \omega(\frac{\partial g(m,t)}{\partial t}, \frac{\partial g(m,t)}{\partial m}\eta)dt, \qquad \eta \in T_m\tilde{U}_h.$$

We conclude that this differential has the form $d\Psi(m)\eta = \tilde{Q}(\frac{\partial g(m,t)}{\partial t}|_{t=0}, \eta)$, where $\tilde{Q}$ is the following bilinear form on $\tilde{U}_h$:

$$\tilde{Q}(\xi_0, \eta) = \int_0^1 \omega(\xi_t, \frac{\partial g(m,t)}{\partial m}\eta)dt.$$

We see, that in any fixed point m of the mapping $\tilde{A}$ we have $\tilde{Q}(\xi_0,\eta) = \frac{1}{2}\omega(\xi_0, d\tilde{A}(m)\eta+\eta)$, $d\Psi(m) = 0$, $d^2\Psi(m)\eta = \tilde{Q}(d\tilde{A}(m)\eta - \eta, \eta) = \omega(d\tilde{A}(m)\eta, \eta)$. In particular, for unperturbed system, for every point $m \in \Lambda$ and tangent vector $\eta \in T_m\Lambda$, we have $Q(*, \eta) = \omega(*, \eta)$.

From the last equality we make the following important conclusion, which we prove below. Namely, we conclude that the "orthogonality" of a vector $\xi_0 \in T_m\tilde{\theta}_m$ to all tangent vectors $\eta \in T_m\tilde{\Lambda}$ (with respect to the bilinear form $\tilde{Q}$) implies $\xi_0 = 0$. Using the dimentional reason, we obtain that the bilinear form $\tilde{Q}$ gives rise to some non-degenerate coupling between tangent spaces of the surface $\tilde{\theta}_m$ and the surface $\tilde{\Lambda} \cap \tilde{\sigma}_m$, for any point $m \in \tilde{\Lambda}$. Thus, on $\tilde{\Lambda}$ there exists single-valued field of directions $\tilde{\eta} \in T_m\tilde{\Lambda}$, $m \in \tilde{\Lambda}$, close to vertical field, and orthogonal (with respest to the form $\tilde{Q}$) to all vectors $\xi_0 \in T_m\tilde{\theta}_m$: $\tilde{Q}(\xi_0, \tilde{\eta}) = 0$.

We see that the function $S = \Psi|_{\tilde{\Lambda}}$ is indeed circular for the unit vector field $\tilde{v}$, tangent to the field $\tilde{\eta}$ of directions. It remaines to prove the conclusion from the formula $Q(*, \eta) = \omega(*, \eta)$, $\eta \in T_m\Lambda$, mentioned above. It is enough to give the proof only for the unperturbed system. Let us notice that the image of the operator $dA(m) - I$ is skew-orthogonal to its kernel (by virtue of symplecticity of mapping A). Consequently, by the nondegeneracy condition on Λ, this image coincides with the skew-orthogonal complement to $T_m(\Lambda \cap \sigma_m)$ in σ_m. From this fact and by construction of the surface θ_m, $m \in \Lambda$, we see that any tangent vector to this surface, which is skew-orthogonal to the subspace $T_m(\Lambda \cap \sigma_m)$, is equal to zero.

From the implicit function theorem we obtain the property 4) from statement 1-C about the smooth dependence on the small parameter of the embedding i, the vector field $\tilde{v}$ and the function ϵS. This implies the smooth dependence on parameter of the function S.

Step 6. Now let $m \in \tilde{\Lambda}$ be any critical point of the function $\Psi|_{\tilde{\Lambda}}$. We'll show that this point is fixed under Pioncare mapping $\tilde{A}$, i. e. the vector of "translation" $\xi_0 = \frac{\partial g(m,t)}{\partial t}|_{t=0}$ vanishes. Indeed, this vector is tangent to the surface θ_m in according to the construction of the submanifold $\tilde{\Lambda}$. From the other hand, we have really assumed that vector ξ_0 is orthogonal to all tangent vectors to $\tilde{\Lambda}$ (with respect to the firm $\tilde{Q}$). Hence, in agreement to the conclusion on the previous step, $\xi_0 = 0$. We have proved the property 2) from statement 1-C and, consequently, theorem 1–A.

Step 7. Let us prove the rest property 3) of statement 1-C, which implies the averaging method on the submanifold. Let us consider the direct product of the extended phase space $M^{2n} \times \mathbb{R}_t$ on the real axis $\mathbb{R}$ with the coordinate ϵ'. On the isoenergy surface of each system with Hamiltonian $H_{\epsilon'} = H + \epsilon' H_1$, $0 \le \epsilon' \le \epsilon$, let us consider the smooth function Ψ, constructed above. By analogy with the second step, we'll "close" 2–chain $\Gamma(m_0, m_1)$ by the chain, composed by orbits of perturbed system on the isoenergy surface, and integrate 2–form $\tilde{\Omega}$ on the obtained chain. Let us subtract from the function S, determined by this way, the unperturbed (identical zero) function, and let $\epsilon \to 0$. To show that the limit equals $-\bar{H}$, we'll apply Stokes' theorem to the integral of the form $\Omega_{\epsilon'} = \omega - dH_{\epsilon'} \wedge dt$ over the boundary of the infinitesimal 3–chain, composed by infinitesimal 2–chains corresponding to systems with Hamiltonians $H_{\epsilon'} = H + \epsilon' H_1$, $0 \le \epsilon' \le \epsilon$. According to Hamilton's principle of the least action, the part of this integral corresponding to the both infinitesimal "sides" of such 3–chain vanishes. Consequently, the required limit of the function S is equal to the limit of the integral of the differential $-d\epsilon' \wedge dH_1 \wedge dt$ of the form $\Omega_{\epsilon'}$ over the 3–chain under consideration, devided by ϵ. But the last limit coincides with the function $-\bar{H}$. This implies property 3) from statement 1-C.

Statement 1-C is completely proved. From this statement one can obtain by standard technique the averaging principle (theorem 2) and statements 1-A, B.

Theorems 1–A, 1*, 2 and statements are proved. It remaines to give a proof of theorem 1–B. ■

Proof of theorem 1–B.

Let us assume, that the quotient manifold $B = \Lambda/S^1$ is locally flat. We claim that in this case the embedding i and the function S (see statement 1-C) can be constructed in such a way that the function S is constant along every orbit of the unperturbed system on Λ. The proof will be analogous to one given above.

Step 1. Let $\sigma_m \subset H^{-1}(h)$, by analogy with the previous proof, be a small cross-section in the isoenergy surface to the orbit passing through the point $m \in \Lambda$. Let us consider (depending on m) Poincaré mapping A of the cross-section σ_m into itself. Further, let us construct the co-kernel of the operator $dA(m) - I$. More exactly, let us denote by $D_m = Im(dA(m) - I)$ the image of this operator. In contrast to the previous proof, let us denote by $\theta_m \subset T_m\sigma_m$ a subspace (but not a surface), having the same dimention as $\Lambda \cap \sigma_m$ and transversal to the subspace D_m in $T_m\sigma_m$. We'll assume that both conctructed transversals: the surface σ_m and the subspace $\theta_m \subset T_m\sigma_m$ depend smoothly on the point $m \in \Lambda$.

Let us consider the obtained fibration over Λ with fibers θ_m, $m \in \Lambda$. Let us notice, that the subspace θ_m is naturaly isomorphic to the co-kernel of the operator $dA(m) - I$, i. e. to the quotient space $E_m = (T_m\sigma_m)/D_m$, $m \in \Lambda$. Let us note some properties of this quotient fibration $E \to \Lambda$ with fiber $E_m, m \in \Lambda$.

A) The rank of the fibration $E \to \Lambda$ is equal to the dimention $\dim B = \dim \Lambda - 1$ of the quotient-manifold B.

B) The natural action of the circle on Λ can be extended to the whole fibration E, with use of the tangent flow of the given system. In other words, the field of subspaces $D_m, m \in \Lambda$, is invariant under the tangent flow, so that the action of the flow on the fibration E is well-defined. Besides, the mapping "over the period" is the identity operator on the fibration E.

C) There exists natural isomorphism (related with coupling mentioned above) between the fibration E and the subfibration of the cotangence fibration $T^*\Lambda$, consisting of all covectors vanishing on the tangent vector to the fiber on Λ. Namelly, under this isomorphism any vector $\xi \in E_m$ is mapped to the covector, which value on the arbitrary vector $\eta \in T_m\Lambda$ equals $\omega^2(\xi, \eta)$. It is clear that such an isomorphism of fibrations is well-defined and commutes with natural action of a circle S^1 on these fibrations.

These properties permit us to realise the following construction. Let us consider some locally flat (affine) connectedness on the quotient manifold B. Let us consider the corresponding connectedness on the cotangent fibration T^*B on B, and "lift" the last connectedness on Λ. As a result we obtain the locally flat connectedness on the subfibration of the cotangent fibration $T^*\Lambda$ described in property C). Finally, using the isomorphism between this subfibration and the fibration E, see C), and the natural isomorphism between the fibration E and the field of subspaces θ_m, $m \in \Lambda$, we obtain the corresponding connectedness on the last field of subspaces θ_m, $m \in \Lambda$.

By construction, we have that this connectedness is flat, and that the action of the circle, see B), on the fibration is the parallel carry with respect to this connectedness. It implies, that for any closed curve in Λ, homotopic to the fiber S^1, the holonomy operator is identical.

Now, by analogy with the previous proof, let us prolonge the described construction on the whole neighborhood of Λ in M. Namely, let us fix a small tubular neigborhood U of the submanifold Λ in M and denote by U_h the intersection of this neighborhood with the isoenergy surface $H^{-1}(h)$. We'll assume that both transversals: the surface σ_m and the subspace $\theta_m \subset T_m\sigma_m$ are determined for all points $m \in U_h$ and smoothly dependent on the point. Let us denote by Θ the fibration over the neighborhood U_h with fibers $\theta_m, m \in U_h$. Let us consider some retraction $U_h \to \Lambda$ of the tubular neighborhood onto Λ. Using this retraction, one can easely construct some locally flat connectedness on the fibration Θ over U_h, such that its restriction on Λ coincides with the connectedness constructed above.

Let us underline ones more the following **special property** of the constructed connectedness on the fibration Θ: the parallel carry along any closed curve in U_h, homotopic to the fibers on Λ, is the identity operator.

Finally, similarly to the previous proof, let us transfer both constructed fields of transversals (σ_m and $\theta_m \subset T_m\sigma_m$) and the connectedness (on the fibration Θ) onto the "perturbed" isoenergy surface $\tilde{U}_h = U \cap \tilde{H}^{-1}(h)$, with use of some diffeomorphism $H^{-1}(h) \to \tilde{H}^{-1}(h)$ close to the identity mapping.

Step 2. For the both systems (i. e. unperturbed and perturbed ones) we here construct in the space $\Theta \supset U_h$ some "extended" dynamical system and "extended" Poincaré mapping $\mathbf{A} : \Theta \to \Theta$. For convenience of notations we'll describe the construction only for the unperturbed system.

Let V be the vector field in U_h, corresponding to the initial dynamical system. We'll define the dynamical system on the "extended" space Θ by presenting the integral orbits of this system. Namelly, a smooth curve $m(t) \in U_h, \xi(t) \in \theta_{m(t)}$ in the space of the fibration Θ we call integral orbit, if:

1) The "projection" $m(t)$ of this curve in U_h has the velocity $dm(t)/dt = V_{m(t)} + \xi(t)$.

2) The field $\xi(t)$ is parallel along this curve $m(t)$.

One can show that these conditions give us the well-defined dynamical system on the space of the fibration Θ. It is clear that the zero section of Θ is invariant under this system, and the restriction of this dynamical system to the zero section coincides with the initial system V on U_h.

Now we can define the extended Poincaré mapping. From each "point" $m \in U_h, \xi \in \theta_m$ of the space Θ we'll move along the integral curve $m(t), \xi(t)$ of the described system ($m(0) = m$, $\xi(0) = \xi$) during a time $\tau(m, \xi)$, close to the period $T|_\gamma$ of the orbit $\gamma \subset \Lambda$ close to $m(t)$. Let the curve $m(t)$ intersects the cross-section σ_m over the time $\tau(m, \xi)$. Then we denote by $\mathbf{A}$ the mapping from the space Θ into itself, which moves the "point" (m, ξ) to the value of the orbit $(m(t), \xi(t))$ in the moment $t = \tau(m, \xi)$. The obtained mapping $\mathbf{A} : \Theta \to \Theta$ we call the *extended Poincaré mapping*. It is clear, that the restriction of this mapping to the zero section of the fibration Θ coincides with the usual Poincaré mapping $A : U_h \to U_h$.

In analogous way, one defines the extended Poincaré mapping $\tilde{\mathbf{A}}$ for the perturbed system.

Let us add to the properties A)–C) of Poincaré mapping (see step 2 of the previous proof) some property of the extended Poincaré mapping $\mathbf{A}$:

D) Let us consider the linearization $\delta\mathbf{A}$ of the extended Pioncare mapping $\mathbf{A}$ in any fixed "point" (m, ξ), $m \in \Lambda$, $\xi = 0$. Let us introduce on Θ local "coordinates" (m, ξ) and consider the representation of the operator $\delta\mathbf{A}$ in terms of "variations" $(\delta m, \delta\xi)$ of these coordinates in the point $(m, 0)$. Then, under this operator, any vector $(\delta m, \delta\xi)$ is moved to some vector having the form $(\delta m + e^T\delta\xi + d, \delta\xi)$. Here $T = T(m) > 0$ denotes the value of period function T in the point $m \in \Lambda$, $d \in D_m$.

Let us consider the image of the submanifold Λ in Θ under the natural embedding as the zero section. We'll denote this image also by Λ. It is clear, that the submanifold $\Lambda \subset \Theta$ is filled by closed orbits of the constructed dynamical system on Θ. Moreover, one can show with use of the property D), that this submanifold is nondegenerate in the sense of the definition 1.

Step 3. Let us define the perturbed submanifold $\tilde{\Lambda} \subset \Theta$ as the set of all "points" $(m, \xi) \in \Theta$, which images under the mapping $\tilde{\mathbf{A}}$ have the form $(m, *)$. It turnes out that this set is filled by closed orbits of the perturbed dynamical system in Θ (which has been constructed in the previous step). In fact, it's enough to show that the set $\tilde{\Lambda}$ coincides with the fixed point set of the extended Poincaré mapping $\tilde{\mathbf{A}}$. But this follows immediately from the definition of $\tilde{\Lambda}$ and the special property of the connectedness on $\tilde{\Theta}$, see step 1.

By the implicit function theorem, for sufficiently small ϵ, the set $\tilde{\Lambda}$ is a smooth submanifold and has the form $\tilde{\Lambda} = i(\Lambda)$, where i is an embedding close to the identity mapping. Indeed, the implicit function theorem is applicable due to the property D) of the extended Poincaré mapping, see step 2.

Since the submanifold $\tilde{\Lambda}$ is filled by closed orbits of the perturbed dynamical system on Θ, one can choose such an embedding i, that images of fibers on Λ under i coincide with the orbits of the dynamical system on Θ. Besides, the time of the motion along these orbits is proportional to the natural parameter on the circle with factor $\frac{\tilde{T}}{2\pi}$, where $\tilde{T}$ is the perturbed function of periods on $\tilde{\Lambda}$. Further, we'll denote by $\tilde{\Lambda}$ also the submanifold $\pi(\tilde{\Lambda})$ in $\tilde{U}_h$, where $\pi : \tilde{\Theta} \to \tilde{U}_h$ is the natural projection. It is evident, that such $\tilde{\Lambda}$ is close to Λ.

Step 4. Here we'll define the smooth function ψ on the quotient manifold $B = \Lambda/S^1$ by an explicit formula. Let $\gamma = \{b_t\} \subset B$ be an arbitrary smooth curve in B. Let us consider the 2–chain $p^{-1}(\gamma)$ in Λ, which is the inverse image of the curve γ, and integrate the symplectic structure on the image of this chain under the embedding i. The obtained real number is the difference of values of the function

ψ on the ends of the curve γ:

$$\psi(b_0) - \psi(b_1) = \iint_{iop^{-1}(\gamma)} \omega.$$

The property C) of Poincaré mapping (see step 2 of the previous proof) implies that the smooth function ψ is global single-valued function on B. For definiteness, let us assume, that this function vanishes in some fixed point b_0 on B. Since under $\epsilon = 0$ the function ψ identically vanishes on B, we'll denote the constructed function by ϵS.

Step 5. Here we give an explicit representation for the differential of the function ψ, and we'll consider this function as determined on the whole Λ. One can show that the differential of the function ψ in an arbitrary point $m \in \Lambda$ has the following form: $d\psi(m)\eta = \tilde{Q}_{m'}(\xi, \pi \circ i_*\eta)$, where $(m', \xi) = i(m) \in \Theta$, ξ is the analog of the vector of "translation", $\tilde{Q}_{m'}$ is the "coupling" between subspaces $\tilde{\theta}_{m'}$ and $T_{m'}(\tilde{\Lambda} \cap \tilde{\sigma}_{m'})$. The bilinear form $\tilde{Q}$ is S^1–invariant under the natural action of the circle, since by definition it is obtained by "averaging" of the symplectic structure. For the unperturbed system we have:

$$Q_m(\xi, \eta) = T(m)\omega(\xi, \eta), \qquad \xi \in \theta_m, \eta \in T_m(\Lambda \cap \sigma_m).$$

From the last equality, using the skew-orthogonality of the considered subspaces, we conclude that the equality $\tilde{Q}_{m'}(\xi, \eta) = 0$ under any $\eta \in T_{m'}\tilde{\Lambda}$ implies $\xi = 0$.

Step 6. Finally, let $m \in \Lambda$ be any critical point of the function ψ. We'll show that its image $i(m)$ is fixed under Pioncare mapping $\tilde{A}$, i. e. the vector of "translation" ξ vanishes. Indeed, we have really assumed that vector ξ is "orthogonal" to all tangent vectors to $\tilde{\Lambda}$ (with respect to the firm $\tilde{Q}$), see step 5. Hence, in agreement to the conclusion on the previous step, $\xi = 0$. This immediately implies the item B) of theorem 1.

Thus, theorem 1 is completely proved. ■

4. The role of constancy for values of the energy and the period function. Some generalization

4.1. Let us point out that theorems 1 and 1* imply the existence of the required number of closed orbits on any isoenergy surface $\tilde{H}^{-1}(h')$ close to the initial one, but not only on the surface $\tilde{H}^{-1}(h)$. It follows from the fact that one can consider the energy value as the additional parameter of perturbation. In any case, theorems 1 и 1* give estimation for the number of closed orbits lying on the same isoenergy surface. But one can obtain the same estimation if we fix the period of required closed orbits, instead of the energy value.

Here we give the analog of theorems 1, 1* and 2 for the case of estimation for the number of closed orbits (of perturbed system) with given period. In contrast to the situation above, we assume here that the period function T on the submanifold Λ is constant (for some orbits it may be not minimal period), for example $T = 1$, but Λ does not necessarily lie on the isoenergy surface. Besides, in this situation Λ can contain critical points of Hamiltonian, so that some orbits on Λ may be equilibrium points. Let us define the Poincaré mapping as the flow $A = g_H^1$ of the system over the time 1 in the whole phase space (but not as the restriction of this flow to the isoenergy surface, in contrast to the situation above). The definition of the nondegeneracy is analogous to the definition 1.

Theorem 3. *Let the submanifold Λ, filled by 1–periodic orbits of the unperturbed system, be compact (without boundary) and nondegenerate. Then:*

A) The number of 1–periodic orbits of the perturbed system is not less than the minimal number of critical orbits of circular function on the manifold Λ. Besides, the number of such orbits, counted with their multiplicities, is not less than the minimal number of critical orbits of Bott circular function on the manifold Λ.

B) Let the fibration of the submanifold Λ by periodic orbits be of special type, namely: either this fibration is trivial; or the base $B = \Lambda/S^1$ of this fibration is locally flat (for example, is diffeomorphic to the multidimensional torus). Then the number of geometrically different 1-periodic orbits of perturbed system is not less than the minimal number of critical points of the smooth function on the quotient manifold B. Besides, the number of such orbits, counted with their multiplicities, is not less than the minimal number of critical points of Morse functions on B.

Theorem 4. *Let the submanifold Λ, filled by 1-periodic orbits of the unperturbed system, be non-degenerate. Let us assume, that the Hamiltonian of the perturbed system depends smoothly on the small parameter ϵ, i. e. has the form (2). Let us consider the averaged perturbation on Λ, i. e. the smooth function $\bar{\mathcal{H}}(m) = \int_0^1 \mathcal{H}(\gamma(m,t))dt$, $m \in \Lambda$, where $\mathcal{H} = H_1|_\Lambda$. Let γ_0 be Bott critical orbit of this function. Then there exists one-parameter family of 1-periodic orbits $\gamma_\epsilon \subset \tilde{H}^{-1}(h)$ of perturbed systems. This family depends smoothly on the parameter of perturbation ϵ, where ϵ is sufficiently small, and γ_ϵ coincides with γ_0 under $\epsilon = 0$.*

Let us show the difference between the averaging principle in theorems 3 and 5. In fact, in theorem 3 we consider not standard averaging over the time, because we do not divide the value of the integral (3) by the period function T. This is essential because the period function T is not necessarily constant on $\Lambda \subset H^{-1}(h)$. But in some of the important cases the period function turnes to be constant.

Statement 2. *Let Σ be a connected submanifold of the phase space (M^{2n}, ω^2), filled by closed orbits of Hamiltonian system with Hamiltonian H. Let the differential $d(H|_\Sigma)$ of the restriction of the function H to Σ does not vanish anywhere on Σ. Let T be some continuous function of period (not necessarily minimal period) of the orbits on Σ. Then this function is the smooth function depending on the value h of Hamiltonian H on Σ. Moreover, the period function has the form $T = dI(h)/dh$, where value $I(h)$ is equal to the integral of the form ω^2 over any 2-chain, composed by periodic orbits of the one-parameter family $\gamma_{h'} \subset \Sigma \cap H^{-1}(h')$, $h_0 \le h' \le h$, where h_0 is some fixed value of the Hamiltonian H.*

In other words, in this situation the period function T equals the derivative with respect to h of the "area" $I(h)$, "swept" by periodic orbits of the form $\gamma_{h'} \subset \Sigma \cap H^{-1}(h')$, where $h_0 \le h' \le h$.

Proof.

There is geometric proof of this statement. In our situation this proof is completely analogous to the standard proof [7], [8], although in this papers it had been done only for periodic systems. We give here the another proof of the first part of the statement, based on the averaging principle. We'll impose some additional condition on Σ, namelly that each submanifold $\Lambda = \Sigma \cap H^{-1}(h)$ is almost everywhere nondegenerate in the sense of definition 1. Let $\tilde{H} = H - \epsilon$ be the perturbed Hamiltonian. Then the perturbed system on the initial energy level $\tilde{H}^{-1}(h)$ is similar to the unperturbed system on the close energy level $H^{-1}(h+\epsilon)$. It is clear, that the averaged perturbation on Λ coincides with the period function T. Thus, from statement 1–A, we conclude that if T is not constant on the submanifold $\Lambda \subset \Omega$, then this submanifold can not be included into the family of analogous submanifolds $\Lambda_\epsilon \subset H^{-1}(h+\epsilon)$, $\Lambda_0 = \Lambda$. This contradict to the condition that the differential of the function $H|_\Sigma$ does not vanish anywhere on Λ.

Thus, under the hypothesis of proposition 1, the period function T will be constant on Λ. Consequently, in this case we can use the usual averaging: $\bar{\mathcal{H}}(m) = \frac{1}{T}\int_0^T \mathcal{H}(\gamma(m,t))dt$, $m \in \Lambda$, so that theorem 2 and statements remain valid.

4.2. Now we pass to formulate some generalizations of theorems above. At first, we consider the sets, filled by closed orbits, which lie on the singular isoenergy surface. Further, we consider the case when the symplectic structure also is perturbed under the perturbation of the system. And in the following section we describe the problem from the celestial mechanics in which the generalised Poincaré theorem is successfully applied, although the unperturbed 2-form of the symplectic structure is degenerate.

By analogy with the situation of theorem 1, let Λ be some invariant closed subset of the isoenergy surface $H^{-1}(h)$, filled by closed orbits. Let the fibration of Λ by closed orbits is periodic, i. e. there exists some continuous positive function T on Λ, such that $g_H^{T(m)}(m) = m$, where g_H^t is the flow of the system over the time t. In other words, every orbit γ on Λ is "closed" over the time $T|_\gamma$. But, in contrast to theorem 1, we'll assume that some orbits on Λ are equilibrium points of the considered Hamiltonian system. In particular, the given isoenergy surface $H^{-1}(h)$ is singular. ■

Definition 5. We call such a singular subset $\Lambda \subset H^{-1}(h)$ *nondegenerate* if all its singularities are of Morse type. More exactly, Λ is called nondegenerate if it satisfies the following conditions:

1) Firstly, all equilibrium points in Λ are Morse critical points of the Hamiltonian function H.

2) Secondly, the complement to the set of equilibrium points in Λ is nondegenerate in the sense of definition 1, see chapter 2.

3) Finally, for any equilibrium point $m \in \Lambda$ and tangent vector $\xi \in T_mM$ to this point, the following equalities under $\epsilon = 0$:

$$dg_H^{T(m)}(m)\xi = \xi \qquad \text{and} \qquad d^2H(m)\xi = \epsilon,$$

implies that vector ξ is tangent to the subset Λ.

Here $d^2H(m)$ denotes Hessian (i. e. "quadratic part") of the function H in the critical point m. Besides, here we call vector $\xi \in T_mM$ *tangent* to the subset Λ if this vector has the form $\xi = dm(t)/dt|_{t=0}$ for some smooth curve $m(t) \subset \Lambda$, $0 \le t < 1$.

Let us apply a small perturbation to the Hamiltonian H, under which the "energy level" becomes regular. In other words, the perturbed isoenergy surface $\tilde{H}^{-1}(h)$ does not contain critical points. Before the formulation of the result, we'll construct the smooth submanifold $\tilde{\Lambda} \subset M$, obtained from Λ by some surgeries near each equilibrium point $m \in \Lambda$, analogous to the usual Morse surgery.

To every equilibrium point $m \in \Lambda$ we attach a sign $\epsilon_m = \pm 1$ in the following natural way. This sign depends on that: if the value h is greater or smaller than the critical value of the perturbed function $\tilde{H}$ in a small neighborhood of the point m. The set $\{m \to \epsilon_m\}$ of these signs we call *the type of the perturbation.* The following construction we'll apply to each equilibrium point $m \in \Lambda$:

1) Firstly, we remove from Λ the point m both with a small ball D_m, having m as the center.

2) Secondly, we consider the smooth submanifold in T_mM, consisting of all tangent vectors $\xi \in T_mM$, satisfying equations above with the corresponding value $\epsilon = \epsilon_m$.

3) And finally, we replace the removing part of Λ in D_m by the smooth "handle" described just now. In other words, we paste this "handle" to $\Lambda \backslash D_m$ by natural one-to-one correspondence of their boundaries.

As the result, we obtain a closed smooth submanifold $\tilde{\Lambda} \subset \Lambda \cup D_m \subset M$ close to Λ. We see from the construction of this manifold, that it depends only on the type of the perturbation. Besides, there exists a natural fibration $\tilde{\Lambda} \to B$ on this submanifold with fiber the circle. Finally, under the natural projection $\tilde{\Lambda} \to \Lambda$ all fibers are transfered to orbits of the initial system on Λ. The obtaned submanifold $\tilde{\Lambda}$ we call *Morse surgery* of the initial set Λ under the given perturbation.

Proposition 1. *Let Λ be a compact nondegenerate singular subset of the isoenergy surface $H^{-1}(h)$, see def. 5, filled by closed orbits. Let $\tilde{\Lambda}$ be the Morse surgery of this set under a perturbation of the given type. Then the estimation above for the number of periodic orbits of the perturbed Hamiltonian system (see theorem 1) remain valid. More exactly, in the estimation of theorem 1 the set Λ (both with fibration on it) should be replaced by the manifold $\tilde{\Lambda}$ with corresponding fibration on it.*

Actually, any submanifold Λ, which doesn't contain equilibrium points, can be considered as the partial case of subsets containing such points. From this point of view, proposition 2 generalizes theorem 1.

REMARK 3. *Partial case of the described situation had been studied by Moser [13] and Weinstein [9], [10], [15]. Namelly, if speaking in our notations, in these works set Λ was an equilibrium point, the submanifold $\tilde{\Lambda}$ was diffeomorphic to sphere, and the perturbation of the Hamiltonian was the identical ϵ.*

4.3. Let us note that in all theorems formulated above we assumed that the symplectic structure ω^2 is fixed on the manifold M^{2n}. Sometimes in applications the symplectic structure changes under perturbation. In turnes out that in some of such cases our theorems remain valid. For example, it is so if either $H^2(\Lambda) = 0$, or $H^1(B) = 0$, or the topology of the fibration (1) satisfies some more weak condition than the previous ones. Here $H^2(\Lambda)$ is the 2-dimentional de Rham cohomology group of the manifold Λ.

But in many cases, important to applications (for example, when Λ is torus), our theorems will not remain valid for arbitrary small perturbations of the symplectic structure. Nevertheless, one can generalise them, imposing some restriction to the perturbations of ω^2, natural for Hamiltonian mechanics. Namelly, using proofs of our theorems, one easily proves the following statement.

Proposition 2. *Theorems 1, 1*, 2, 3, 4 remain valid, if under the perturbation of the system we permit C^1-small perturbation of the symplectic structure ω^2, such that the cohomology class of ω^2 does not shange under perturbation. Moreover, the last condition can be replaced by the following (more weak) condition, called "preserving of the center of mass", see chapter 3 (and also works [12], [18], [24]). Namelly, this condition means that for any closed curve γ in B the integral of the form $\tilde{\omega}^2$ over the 2-chain $p^{-1}(\gamma)$ vanishes, see [24]. Let us note that for the unperturbed 2-form ω^2 this condition is always satisfied.*

Let us underline ones more that we don't impose any restriction to the initial symplectic structure ω^2, since all of required conditions will be satisfied automatically. Moreover, it turnes out that even if the unperturbed structure ω^2 is degenerate, but the perturbed one is "admissible", then our theorems remain valid. For example, such situation appears in study of relatively periodic motions of planet-satellite system, which we discuss in the following section. The similar result for planet system with double planets is discussed in the author's work [28]. Another examples illustrating such an application of the generalized geometric Poincaré theorem are S. V. Bolotin's and A. I. Neishtadt's results on the existence of periodic motions of a charged particle in multidimentional strong magnetic field (or periodic motion of mechanical system under large gyroscopic forces).

5. Periodic motions of a planet-satellite system

Let us consider the problem of celestial mechanics, about the motions of the system of $N+1$ material points in Euclidean plane, attracting to each other due to Newton's law. We ussume that one of these points M_0 (sun) is central point of mass 1. Further, we ussume that all other points $M_1, \dots, M_N$ are decomposed into two groups: n "planets" with masses of order μ, and $N-n$ "satellites" with masses of smaller order $\mu\nu$, where μ and ν are small parameters. Let the attraction between points be Newtonian with the potential

$$U = -\sum_{0 \le i < j \le N} \frac{m_i m_j}{r_{ij}}.$$

Here m_i denotes the mass of the point M_i, and $r_{ij} = |M_j - M_i|$, $0 \le i, j \le N$, are mutually distances between the points. Thus, the motion is described by the following equations: $m_i d^2 M_i/dt^2 = -\partial U/\partial M_i$, $0 \le i \le N$.

Let us pass to the frame of reference rotating with constant angular velocity, and set up the problem of looking for T-periodic motions of the described gynamical system (with respect to the rotating coordinate system). Such motions are specified by the following property: over the time

$T > 0$ the configurations together with velocities of all points of the system can be obtained from the initial ones by the rotation over the same angle $\alpha (mod\, 2\pi)$ around the origin.

Definition 6. Such a motion we call *relatively periodic* with *parameters* T, α. Let us consider the other motion which can be constructed from the initial one by the following way. At first, one can move the origin of the time axis, and then one can rotate the plane around the origine over some angle. It is clear that all the motions obtained by such a way remain relatively periodic with the same parameters T, α. All of these motions we'll consider as the same relatively periodic motion. Let us note that the set of these motions is diffeomorphic to the two–dimentional torus.

Put $\epsilon = 2\pi/T$. We'll assume that the positive numbers μ, ν and ϵ are small, functionally independent parameters. Another parameters of our problem will be the fixed number α and the set of $N-1$ integer numbers which orders will be dependent on ϵ. Let us describe these parameters more precisely.

At first, let us introduce more convenience numeration of points. Let us remind, that we implicitly assume that each satellite is associated with some planet. For notation of such a correspondence we introduce the new numeration of points with use of two indexes: M_{ij}, $1 \le i \le n$, $0 \le j \le n_i$. Here M_{00} is the sun,

$$M_{i0}, \qquad 1 \le i \le n, \qquad \text{and} \qquad M_{ij}, \qquad 1 \le j \le n_i,$$

are planets and their satellites, respectively. Satellites with their planet compose the so-called i–th satellite system. We'll also designate the masses of planets as μm_{i0}, and masses of satellites as $\mu\nu m_{ij}$, where $m_{ij} = const > 0$, $1 \le i \le n$, $0 \le j \le n_i$.

Let us note, that we study special solutions of the described $N+1$ body problem. Namely, we assume that the mutual distances between each planet and its satellites have the order 1, while the mutual distances between the sun and all planets have the more greater order $R >> 1$, which is computed by the formula $\epsilon = 1/\sqrt{\mu R^3}$, and consequently, it is automatically large. In accordind to this, we introduce the "relative coordinates". Namelly, from each planet we draw radius vectors

$$\mathbf{y}_{ij} = M_{ij} - M_{i0}, \qquad 1 \le j \le n_i,$$

to all its satellites. Besides, for each satellite system we consider the normalized radius vector

$$\mathbf{x}_i = (C_i - M_{00})/R, \qquad 1 \le i \le n,$$

drawing from the sun to the center of mass $C_i = (m_{i0}M_{i0} + \nu \sum_{j=1}^{n_i} m_{ij}M_{ij})/(m_{i0} + \nu \sum_{j=1}^{n_i} m_{ij})$ of this satellite system. (Actually, it should be considered one more radius vector: drawing from the origin to the center of mass C of the whole system of points. But since C is "cyclic variable", one can either eliminate it, or pass to the frame with origine in C, which is equivalent.)

Further, we assume that "months" are much smaller than "years". More exactly, after rescaling the time, we assume that "mean frequencies" of rotations of radius vectors $\mathbf{y}_{ij}$ are "fast" of order 1 (months), and "mean frequencies" of rotations of radius vectors $\mathbf{x}_i$ are "slow" of order ϵ (years). Let us consider the set of "mean relative frequencies" of rotations of described radius vectors under some relatively periodic motion. It is clear that this set is proportional to some set of integer numbers, i. e. this set is maximal resonance. Indeed, the set of "mean frequencies" is of the form

$$\begin{aligned} \omega_i &= \omega_1 + k_i\epsilon \qquad (1 \le i \le n) \\ \Omega_{ij} &= \omega_1 + K_{ij}\epsilon \qquad (1 \le j \le n_i), \end{aligned} \tag{4}$$

where $\omega_1 = \epsilon\alpha/2\pi$, and k_i, K_{ij} are some integer numbers. Namely, each of these numbers is the number of rotations of the corresponding radius vector over the time T (with respect to the potating frame). Here we assume, without loss of generality, that $k_1 = 0$, so that the real number $\alpha = \omega_1 T$ equals the whole angle on which the first planets rotates over the time T. Thus, all the integer numbers $k_1 = 0, k_2, \dots, k_n$, corresponding to slow frequencies (of planets) are bounded, while the integer numbers K_{ij} corresponding to fast frequencies (of satellites) have the order $1/\epsilon$:

$$k_1 = 0, k_2 \approx \dots \approx k_n \approx 1, \qquad K_{ij} \approx 1/\epsilon, \qquad 1 \le i \le n, 1 \le j \le n_i. \tag{5}$$

Let us consider as the unperturbed system the set of N independent Kepler problems:

$$\begin{cases} d^2\mathbf{x}_i/dt^2 = -\epsilon^2 \mathbf{x}_i/|\mathbf{x}_i|^3 & (1 \le i \le n) \\ d^2\mathbf{y}_{ij}/dt^2 = -m_i \mathbf{y}_{ij}/|\mathbf{y}_{ij}|^3 & (1 \le j \le n_i). \end{cases} \tag{6}$$

It is easily to see that under $\alpha \not\equiv 0 \pmod{2\pi}$ relatively periodic motions of unperturbed problem with given set of mean frequencies (4) correspond to circular motions of Kepler problems with this set of frequencies. Consequently, orbits of these motions compose the N–dimentional torus in the phase space.

Let us fix the sufficiently small value of parameter ϵ. Let ω_i, Ω_{ij}, be any set of real numbers, having the form (4), (5). Let us specify values of parameters T, α in definition 6, setting $T = 2\pi/\epsilon$, $\alpha = \omega_1 T$.

Proposition 3. *A) Under any $\alpha \not\equiv 0 \pmod{2\pi}$, if μ, ν are sufficiently small, then there exist at least 2^{N-2} (counted with their multiplisities) relatively periodic motions of planet-satellite system, with parameters T, α. Besides, among these motions there are at least $N-1$ geometrically different motions.*

B) Under each of these motions, the mean values of the radius vectors $\mathbf{x}_i = (C_i - M_0)/R$ and $\mathbf{y}_{ij} = M_{ij} - M_{i0}$ are exactly equal to ω_i, Ω_{ij}, and the motion of each of these radius vectors is close to the circular motion with the same frequency of the corresponding Kepler problem (6).

Here the nearness means as follows. The motion $\mathbf{x}_i = \mathbf{x}_i(\tau/\epsilon)$ differs from the circular motion with frequency (ω_i/ϵ) by the value of the order ν/R^2. The motion $\mathbf{y}_{ij} = \mathbf{y}_{ij}(t)$ differs from the circular motion by the value of the order ϵ^2. Here we imply that the corresponding coordinate functions are close with respect to C^1–norm over the "own time" τ and t respectively.

Proof.

Let us fix some value $\epsilon > 0$, and consider the unperturbed problem, corresponding to $\mu = \nu = 0$. One shows that such an unperturbed problem is decomposed into n Kepler problems, corresponding to planets, and $N - n$ problems corresponding to satellites and coinciding with the unperturbed Hill problem. More exactly, the unperturbed system has the form

$$\begin{cases} d^2\mathbf{x}_i/dt^2 = -\epsilon^2 \mathbf{x}_i/|\mathbf{x}_i|^3 & (1 \le i \le n) \\ d^2\mathbf{y}_{ij}/dt^2 = -m_i \mathbf{y}_{ij}/|\mathbf{y}_{ij}|^3 + \epsilon^2 m_i/m_{ij} \partial P/\partial \mathbf{y}(\mathbf{x}_i, \mathbf{y}_{ij}) & (1 \le j \le n_i), \end{cases}$$

where the function $P(\mathbf{x}, \mathbf{y}) = \dfrac{3\langle \mathbf{x}, \mathbf{y}\rangle^2 - |\mathbf{x}|^2|\mathbf{y}|^2}{2|\mathbf{x}|^5}$, multiplied by ϵ^2 is the unperturbed "potential of the action of sun on a satellite". We obtain, that, under small ϵ, variables $\mathbf{x}$ and $\mathbf{y}$ are automatically slow and fast, respectively.

The rest of the proof is based on the geometric Poincaré theorem, namelly, on theorem 3–B. The difficulty lies in the fact that this theorem can not be applied directly, since the unperturbed system is not Hamiltonian. In particular, we must verify the condition "preservation of the center of mass". But in our case it is evident, because the perturbed symplectic structure is the standard one on the cotangent space of the configuration space, and automatically is exact. More detailed proof the author hopes to public in the nearest future. ■

REMARK 4. *In the case of the planet system without satellites ($n_i = 0$) the analogous result had been obtained by Krasinskii in the work [11]. In this work Krasinskii not only proved the existence of the* *relatively periodic solutions, but at first he also found their "generatives" (namely, 2^{N-2} solutions, for which there exists such a moment when all points lie on the same line). But here he used the additional reason about the "reversibility" of the $N+1$ body problem. And, following Poincaré technique [1], Krasinskii partially proved the stability of one of these solutions. Let us note also, that in the case of the system with double planets, the smallness condition on the parameter ν is unessential (see author's work [28]). If, in addition, the number $N+1$ of points equals 3, i. e. the system is of the type Sun–Earth–Moon, then the statement remains valid without smallness condition on parameters μ and ν. The corresponding solutions are well known and called generalized Hill solutions.*

6. Non-Hamiltonian case

In conclusion, we give formulations of our main results for dynamical systems of general type (not necessarily Hamiltonian). Let us consider on an arbitrary smooth manifold M^n the dynamical system, defined by some vector field V. We assume, that in the phase space M^n there exists a smooth submanifold Λ, which does not contain singular points and is filled by periodic orbits of the system. We also assume, that the fibration (1) of this submanifold by periodic orbits is periodic, i. e. on Λ there is a continuous function T, which is equal to the (not necessarily minimal) period of the orbits.

By analogy with Hamiltonian case, for an arbitrary point $m \in \Lambda$ we consider a cross-section $\sigma_m \ni m$ and define the Poincaré mapping $A : \sigma_m \to \sigma_m$ of this cross-section onto itself. Namely, this mapping is defined by the flow of the vector field V over the time close to the value $T(m)$ of period function in point m. Without loss of generality, we'll assume later that the period function T on Λ is the identical 1.

We assume, that the submanifold Λ is nondegenerate in the sense of definition 1, i. e. in each point $m \in \Lambda$ the space of all tangent vectors, fixed under the tangent mapping $dA(m)$, coincides with the tangent space to the surface $\Lambda \cap \sigma_m$ in the point m. As in proof of theorem 1–B, we consider on Λ a field of the subspaces $E_m = (T_m\sigma_m)/Im(dA(m) - I)$, $m \in \Lambda$, which really are co-kernels of the operators $dA(m) - I$, $m \in \Lambda$. The obtained field E of the quotient spaces is some linear fibration $E \to \Lambda$ over the submanifold Λ. It is easily to see that this fibration does not depend on the surfaces σ_m. More exactly, the fibration E is well-defined up to a natural fiber isomorphism of fibrations.

As it was noticed above, see section 3, the natural action of the circle on Λ can be extended to the whole space E by the evident way. Namely, this action is defined with use of the tangent flow of the given dynamical system. In particular, the mapping "over the period" is the identity operator on the fibration E. This implies that the fibration E can be "projected" onto some fibration over the quotient $B = \Lambda/S^1$. The resulting fibration we'll denote by pE.

From the nondegeneracy condition on Λ we obtain, firstly, that the rank of the linear fibration pE over B equals the dimention $\dim B = \dim \Lambda - 1$ of this quotient manifold, and, secondly, that the total space pE of this fibration is orientable. Consequently, the Euler class $e(pE)$ of this fibration is an integer number, which is called Euler number of the fibration pE. Let us remind the "geometric" sense of the Euler number. It is equal to the algebraic number of zeros for any general section $S : B \to pE$ of the linear fibration pE over B.

Let us consider on the phase space the perturbed vector field $\tilde{V}$, which is close to V with respect to the norm C^1. The following theorem is analogous to theorem 1.

Theorem 5. *Let the submanifold $\Lambda \subset M$ be filled by the closed orbits of the unperturbed system. Let this submanifold be closed (i. e. compact and without boundary), and nondegenerate. Let us ussume that:*

A) either the periodic fibration (1) of this submanifold by periodic orbits is trivial,

B) or the constructed above fibration pE over the quotient manifold $B = \Lambda/S^1$ possesses locally flat connectedness.

Then the number of geometrically different closed orbits of perturbed system is not less than the minimal number of zeros of section $S : B \to pE$ of the linear fibration pE over B. Besides, the number of such orbits, counted with their multiplicities, is not less than the minimal number of zeros of the general section of the fibration pE.

In particular, if the Euler number $e(pE)$ of the fibration pE does not vanish, then the perturbed system has at least one closed orbit. Besides, the number of such orbits with their multiplicities is at least equal to the absolute value $|e(pE)|$ of the Euler number of this fibration.

Let us now describe the averaging method on a submanifold for arbitrary dynamical systems. Let the perturbed vector field has the form

$$\tilde{V} = V + \epsilon V_1 + o(\epsilon), \tag{7}$$

where ϵ is a small parameter. Let us consider the restriction $\mathcal{V} = V_1|_\Lambda$ of the perturbation V_1 to Λ, and define the averaging $\bar{\mathcal{V}}$ of the obtained vector field by the formula

$$\bar{\mathcal{V}} = \int_0^1 g_*^t|_B \mathcal{V} dt, \qquad \text{where} \qquad g_*^t|_B \mathcal{V} = dg^{-t}\mathcal{V}(g^t|_\Lambda),$$

where g^t denotes the flow along the vector field V over the time t. Let us remind, that we assume without loss of generality, that the period function T on Λ is the identical 1. The obtained vector field $\bar{\mathcal{V}}$ on Λ we project onto the quotient fibration $E = \cup_{m\in\Lambda}(T_m\sigma_m)/Im(dA(m) - I)$ with use of the natural projections $\pi_m : T_m\sigma \to E_m$, $m \in \Lambda$. One can show that the obtained section $\pi\bar{\mathcal{V}}$ of the fibration E is invariant under the natural action of the circle S^1 on E. Hence, this section can be "projected" onto some section $p\pi\bar{\mathcal{V}}$ of the linear fibration pE. Let us call this section *the averaged perturbation,* since it is analogous to the averaged perturbation $\bar{\mathcal{H}}$ in Hamiltonian case. One proves that this definition of the averaged perturbation is well-defined, i. e. it does not depend on the surfaces σ_m.

The following theorem is analogous to theorem 2 for dynamical systems of general type (not necessarily Hamiltonian).

Theorem 6. *Let the submanifold $\Lambda \subset M$, filled by closed orbits of the unperturmed system, is nondegenerate, but not necessarily compact. Let us assume, that the perturbed vector vield $\tilde{V}$ depends smoothly on a small parameter ϵ, i. e. it has the form (7). And let $b_0 \in B$ be a nondegenerate zero of the averaged perturbation $p\pi(\bar{\mathcal{V}})$. Let $\gamma_0 = p^{-1}(b_0) \subset \Lambda$ be the orbit of the unperturbed system, corresponding to the point b_0. Then there exists one-parameter family of closed orbits γ_ϵ of the perturbed system. This family depends smoothly on the parameter of perturbation ϵ under small ϵ, and γ_ϵ coincides with γ_0 under $\epsilon = 0$.*

We see that in the case of dynamical systems of general type, i. e. not necessarily Hamiltonian, one must know the topology of some linear fibration pE over quotient manifold B. This fibration depends only on the unperturbed system. It turnes out that in many important cases this fibration is fiber-isomorphic to the tangent fibration T_*B of the manifold B.

Statement 3. *In the following cases the linear fibration pE over the manifold $B = \Lambda/S^1$ constructed above is fiber-isomorphic to the tangent fibration T_*B of this manifold B:*

1) when the unperturbed system is either Hamiltonian, or it is the restriction of some Hamiltonian system to the regular isoenergy surface $M_h = H^{-1}(h)$;

2) when the unperturbed system is gradient, i. e. $V = \bigtriangledown F$, where F is some smooth function on M;

3) when for each point $m \in \Lambda$ the multiplicity of 1 in the spectrum of the operator $dA(m)$ is exactly equals to the dimentional of the quotient manifold B.

In particular, in all of these cases Euler number $e(pE)$ of the fibration pE coincides with the Euler characteristic $\chi(B)$ of the quotient manifold B. Thus, if the Euler characteristic of manifold B for such a system does not vanish, then, according to theorem 5, the perturbed system has at least one closed orbit.

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ОБОБЩЕНИЕ ГЕОМЕТРИЧЕСКОЙ ТЕОРЕМЫ ПУАНКАРЕ В СЛУЧАЕ МАЛЫХ ВОЗМУЩЕНИЙ

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Рассмотрим динамическую систему, в фазовом пространстве которой имеется замкнутое подмногообразие, сплошь заполненное замкнутыми траекториями. Исследуется вопрос, сколько и какие из этих траекторий сохранятся, лишь слегка продеформировавшись, при малом возмущении системы. В случае возмущений общего вида ответ дается в терминах усреднения возмущения по замкнутым траекториям исходной системы. Основным результатом данной работы является следующая теорема. Пусть на симплектическом многообразии (M^{2n}, ω^2) задана гамильтонова система с гамильтонианом H. Пусть $\Lambda \subset H^{-1}(h)$ — замкнутое невырожденное подмногообразие, сплошь заполненное замкнутыми траекториями этой системы. Тогда для любой функции $\tilde{H}$, C^2–близкой к функции H, система с гамильтонианом $\tilde{H}$ имеет не менее двух замкнутых траекторий на поверхности $\tilde{H}^{-1}(h)$. При этом, если расслоение Λ на замкнутые траектории тривиально, либо база $B = \Lambda/S^1$ этого расслоения обладает плоской аффинной связностью, то число таких траекторий не меньше, чем минимальное число критических точек гладкой функции на фактор-многообразии B.