

**Chapter 1 Some Necessary Results from
Graph Theory and Geometry**

- 1 Metric and Topological Spaces / 1
 - 1.1 Metric Spaces
 - 1.2 Topological Spaces
 - 1.3 Mappings
 - 1.4 Connectedness, Separability, Compactness
 - 1.5 Homotopy
 - 1.6 Cell Complexes
 - 1.7 Triangulation
 - 1.8 Fundamental Group
 - 1.9 Locally Euclidean Spaces and Manifolds
- 2 Graphs: Topological Approach / 24
 - 2.1 Topological Graphs and Equivalence
 - 2.2 Routes, Paths, Cycles
 - 2.3 Subgraphs, Frames
 - 2.4 Weighted Graphs, Minimal Weight Frames
- 3 Smooth Manifolds / 28
 - 3.1 Smooth Structure, Smooth Mappings
 - 3.2 Tangent Vectors and Vector Fields
 - 3.3 Tensors
 - 3.4 Differential Forms
 - 3.5 Integral of Differential Forms
 - 3.6 Riemannian Metric on Smooth Manifolds
 - 3.7 Covariant Differentiation
 - 3.8 Geodesics
 - 3.9 Curvature of the Riemannian Manifold
 - 3.10 The First and Second Variations of Curve Length
 - 3.11 Geodesic Deformations and Jacobi Fields
- 4 Networks on Manifolds / 56
 - 4.1 Parametric Networks, Reduced Parametric Networks, Degeneracy Components

- 4.2 Smooth, Piecewise-Smooth, Embedded and Immersed Parametric Networks
- 4.3 The Boundary of a Network: Closed Networks
- 4.4 Network Equivalence
- 4.5 The Length of a Network in a Riemannian Manifold
- 4.6 Topological Space of Parametric Networks
- 4.7 Deformations of Parametric Networks
- 4.8 The First Variation Formula for Geodesic Networks
- 4.9 The Second Local Geodesic Variation of Immersed Parametric Networks
- 4.10 Two-dimensional Surface: Euler's Formula
- 4.11 Planar Graphs: Pontryagin-Kuratowski Theorem

5 Networks and Traces / 67

- 5.1 Space of Traces
- 5.2 Boundary of a Trace: Closed Traces
- 5.3 Length of a Trace
- 5.4 Canonical Representative
- 5.5 Trace Deformations
- 5.6 Local Structure of Traces

6 Convex Function Properties / 79

- 6.1 Definition of Convex Functions
- 6.2 Support Plane and Subgradient
- 6.3 Convexity of the Length Function on Euclidean Space
- 6.4 Minimax Theorem for Families of Concave Functions

Chapter 2 The Steiner Problem and Its Modifications

87

1 Families of Networks / 89

- 1.1 Types of Minimality
- 1.2 Traditional Families of Networks
- 1.3 Important Nonclassical Families of Networks

2 Existence Theorems / 97

- 2.1 Parametric Networks with a Fixed Boundary
- 2.2 Closed Parametric Networks
- 2.3 Traces with Fixed Boundary
- 2.4 Closed Traces

3 Questions of Uniqueness / 102

- 3.1 Closed Networks
- 3.2 Networks with Boundaries

Chapter 3 Local Structure of Minimal Networks

105

1 Local Structure of Minimal Parametric Networks / 105

- 1.1 Local Structure of Immersed Minimal Networks
- 1.2 Weighted Parametric Networks
- 1.3 General Case
- 1.4 Proof of Uniqueness Theorem

2 Local Structure of Minimal Traces / 119

- 1 The Minimal Realization Problem for Networks of Given Topology / 124
 - 1.1 Trees
 - 1.2 Nondegenerate Networks with Cycles
 - 1.3 Steiner Networks of General Type
- 2 Convex Minimal Realization of Networks of Given Topologies / 129
 - 2.1 Twisting Number of 2-Trees
 - 2.2 Fundamental Cycles of Nondegenerate Minimal Networks with Convex Boundaries: Trivial Networks
 - 2.3 Twisting Number of Trivial Networks
- 3 k -Convex Minimal Realizations of Networks / 138
- 4 The Minimal Realization Problem for Special Boundary Sets / 139
- 5 Bifurcations Problem / 140

Chapter 5 Global Minimal Networks on the Plane

- 1 Minimal Steiner Trees / 157
 - 1.1 Steiner Hull
 - 1.2 Nondegenerate Steiner Trees
 - 1.3 Hexagonal Coordinate System
 - 1.4 Minimal Steiner Trees with Boundaries of Special Type
- 2 Minimal Spanning Trees / 160
 - 2.1 Delaunay Triangulation and the Voronoi diagram
 - 2.2 Euclidean Traveling Salesman Problem
- 3 Steiner Ratio / 166
 - 3.1 Three-Points Case
 - 3.2 Gilbert-Pollak Conjecture and Minimax Problems
 - 3.3 Four-Points Case
 - 3.4 Minimal Hexagonal Trees
 - 3.5 Inner Spanning Trees

Chapter 6 Planar Local Minimal Networks with Convex Boundaries

- 1 Complete Classification of Minimal 2-Trees with Convex Boundaries / 179
 - 1.1 Tiling Realization of 2-trees whose Twisting Number Does Not Exceed 5
 - 1.2 Tilings and Their Properties
 - 1.3 Structural Elements of Skeletons from \mathcal{WP}_5
 - 1.4 Reduction and Antireduction
 - 1.5 Profiles and their Properties
 - 1.6 Classification Theorem for Skeletons from \mathcal{WP}_5

- 1.7. Location of the Growths of Tilings from WP_5 on Their Skeletons
- 1.8. Theorem of Realization
- 2 Nondegenerate Minimal Networks with Convex Boundaries: Cyclical Case / 240
 - 2.1. Dual Complexes
 - 2.2. The Twisting Number of the Contour Edges of the Dual Complex of a Trivial Network
 - 2.3. Kernels of a Dual Complex
 - 2.4. Tiling Realization of a Trivial Network whose Twisting Number Does Not Exceed 5
 - 2.5. Description of a Tiling of General Form
 - 2.6. Skeletons from P_5
 - 2.7. Location of Growths on Skeletons of Tilings from P_5
 - 2.8. Degenerate Networks

Chapter 7 Planar Local Minimal Networks with Regular Boundaries

273

- 1 Rains / 273
- 2 Construction of a Minimal Realization of a Snake on an Arbitrary Set / 275
 - 2.1. The Characteristic Arc
 - 2.2. The Characteristic Arc in the Case of Three Points
 - 2.3. The Characteristic Arc in the General Case
 - 2.4. The Totally Characteristic Arc
- 3 An Existence Theorem for a Snake Spanning a Regular n -gon / 285
 - 3.1. Characteristic Arcs and Characteristic Triangles
 - 3.2. Proof of the Realization Theorem of a Snake
- 4 Skeletons and Regular n -gons / 292
 - 4.1. Location of Ends
 - 4.2. The Structure of Ends
 - 4.3. The Structure of Profiles
- 5 Criterion for the Existence of an RM -Realization of a Nonlinear Skeleton / 298
- 6 RM -Realization of Skeletons with Three Ends / 303
- 7 Appendix / 309

Chapter 8 Closed Minimal Networks on Closed Surfaces of Constant Curvature

313

- 1 Minimal Networks on Surfaces of Constant Positive Curvature / 315

- 1.1. Closed Minimal Networks on S^2
- 1.2. Closed Minimal Networks on $\mathbb{R}P^2$
- 2 Classification of Closed Minimal Networks on Flat Tori / 321
 - 2.1. Description of Flat Metrics on a Two-Dimensional Torus
 - 2.2. Flat Tori Translation Groups, Lattices, and Universal Coverings
 - 2.3. Net Geodesics
 - 2.4. Partitions of the Plane into Hexagons
 - 2.5. The Type of a Network
 - 2.6. The Characteristic Triangle
 - 2.7. Primary Results
- 3 Classification of Closed Minimal Networks on Flat Klein Bottles / 353
 - 3.1. Description of Flat Metrics on a Klein Bottle
 - 3.2. The Universal Covering of a Flat Klein Bottle
 - 3.3. The Covering of a Flat Klein Bottle by a Flat Torus
 - 3.4. Networks of the First Discrete Type
 - 3.5. Networks of the Second Discrete Type
 - 3.6. Regular Networks
 - 3.7. Classification Theorems
- 4 Closed Networks on Two-Dimensional Surfaces of Negative Curvature / 368
 - 4.1. Metric Restrictions on the Structure of Closed Networks: The Gauss–Bonnet Theorem
 - 4.2. Examples of Closed Minimal Networks on Surfaces of Negative Curvature

Chapter 9 Minimal Networks in Other Spaces

373

- 1 The Case of Polyhedra / 374
 - 1.1. Developments
 - 1.2. Local Geodesics
 - 1.3. Local Structure of Minimal Networks on Polyhedra
 - 1.4. The Gauss–Bonnet Theorem for Polyhedra
 - 1.5. Metric and Topological Restrictions on the Structure of Closed Minimal Networks
 - 1.6. The Case of Convex Polyhedra
 - 1.7. The Case of Regular Polyhedra
- 2 Classification of Closed Minimal Networks on Regular Tetrahedra / 391
 - 2.1. The Branching Covering of Tetrahedra by the Plane and by the Flat Torus
 - 2.2. Regular Networks
 - 2.3. Classification Theorems
- 3 Networks on the Lobachevskian Plane / 398

References

401

Index

405