Preface to the First Edition v

CHAPTER 1
Geometry in Regions of a Space. Basic Concepts 1
§1. Co-ordinate systems 1
1.1. Cartesian co-ordinates in a space 2
1.2. Co-ordinate changes 3
§2. Euclidean space 9
2.1. Curves in Euclidean space 9
2.2. Quadratic forms and vectors 14
§3. Riemannian and pseudo-Riemannian spaces 17
3.1. Riemannian metrics 17
3.2. The Minkowski metric 20
§4. The simplest groups of transformations of Euclidean space 23
4.1. Groups of transformations of a region 23
4.2. Transformations of the plane 25
4.3. The isometries of 3-dimensional Euclidean space 31
4.4. Further examples of transformation groups 34
4.5. Exercises 37
§5. The Serret–Frenet formulae 38
5.1. Curvature of curves in the Euclidean plane 38
5.2. Curves in Euclidean 3-space. Curvature and torsion 42
5.3. Orthogonal transformations depending on a parameter 47
5.4. Exercises 48
§6. Pseudo-Euclidean spaces 50
6.1. The simplest concepts of the special theory of relativity 50
6.2. Lorentz transformations 52
6.3. Exercises 60

CHAPTER 2
The Theory of Surfaces 61
§7. Geometry on a surface in space 61
7.1. Co-ordinates on a surface 61
7.2. Tangent planes 66
7.3. The metric on a surface in Euclidean space 68
7.4. Surface area 72
7.5. Exercises 76
§8. The second fundamental form 76
8.1. Curvature of curves on a surface in Euclidean space 76
8.2. Invariants of a pair of quadratic forms 79
8.3. Properties of the second fundamental form 80
8.4. Exercises 86
§9. The metric on the sphere 86
§10. Space-like surfaces in pseudo-Euclidean space 90
10.1. The pseudo-sphere 90
10.2. Curvature of space-like curves in $\mathbb{R}_1^3$ 94
§11. The language of complex numbers in geometry 95
11.1. Complex and real co-ordinates 95
11.2. The Hermitian scalar product 97
11.3. Examples of complex transformation groups 99
§12. Analytic functions 100
12.1. Complex notation for the element of length, and for the differential of a function 100
12.2. Complex co-ordinate changes 104
12.3. Surfaces in complex space 106
§13. The conformal form of the metric on a surface 109
13.1. Isothermal co-ordinates. Gaussian curvature in terms of conformal co-ordinates 109
13.2. Conformal form of the metrics on the sphere and the Lobachevskian plane 114
13.3. Surfaces of constant curvature 117
13.4. Exercises 120
§14. Transformation groups as surfaces in N-dimensional space 120
14.1. Co-ordinates in a neighbourhood of the identity 120
14.2. The exponential function with matrix argument 127
14.3. The quaternions 131
14.4. Exercises 136
§15. Conformal transformations of Euclidean and pseudo-Euclidean spaces of several dimensions 136

CHAPTER 3
Tensors: The Algebraic Theory 145
§16. Examples of tensors 145
§17. The general definition of a tensor 151
17.1. The transformation rule for the components of a tensor of arbitrary rank 151

17.2. Algebraic operations on tensors 157
17.3. Exercises 161
§18. Tensors of type $(0, k)$ 161
18.1. Differential notation for tensors with lower indices only 161
18.2. Skew-symmetric tensors of type $(0, k)$ 164
18.3. The exterior product of differential forms. The exterior algebra 166
18.4. Skew-symmetric tensors of type $(k, 0)$ (polyvectors). Integrals with respect to anti-commuting variables 167
18.5. Exercises 170
§19. Tensors in Riemannian and pseudo-Riemannian spaces 170
19.1. Raising and lowering indices 170
19.2. The eigenvalues of a quadratic form 172
19.3. The operator $*$ 174
19.4. Tensors in Euclidean space 174
19.5. Exercises 175
§20. The crystallographic groups and the finite subgroups of the rotation group of Euclidean 3-space. Examples of invariant tensors 176
§21. Rank 2 tensors in pseudo-Euclidean space, and their eigenvalues 197
21.1. Skew-symmetric tensors. The invariants of an electromagnetic field 197
21.2. Symmetric tensors and their eigenvalues. The energy-momentum tensor of an electromagnetic field 202
§22. The behaviour of tensors under mappings 205
22.1. The general operation of restriction of tensors with lower indices 205
22.2. Mappings of tangent spaces 207
§23. Vector fields 208
23.1. One-parameter groups of diffeomorphisms 208
23.2. The exponential function of a vector field 210
23.3. The Lie derivative 211
23.4. Exercises 215
§24. Lie algebras 216
24.1. Lie algebras and vector fields 216
24.2. The fundamental matrix Lie algebras 218
24.3. Linear vector fields 223
24.4. Left-invariant fields defined on transformation groups 225
24.5. Invariant metrics on a transformation group 227
24.6. The classification of the 3-dimensional Lie algebras 229
24.7. The Lie algebras of the conformal groups 231
24.8. Exercises 236

CHAPTER 4
The Differential Calculus of Tensors 238
§25. The differential calculus of skew-symmetric tensors 238
25.1. The gradient of a skew-symmetric tensor 238
25.2. The exterior derivative of a form 241
25.3. Exercises 247
§26. Skew-symmetric tensors and the theory of integration 248
26.1. Integration of differential forms 248

26.2. Examples of integrals of differential forms 254
26.3. The general Stokes formula. Examples 259
26.4. Proof of the general Stokes formula for the cube 267
26.5. Exercises 269
§27. Differential forms on complex spaces 270
27.1. The operators d' and d'' 270
27.2. Kählerian metrics. The curvature form 273
§28. Covariant differentiation 275
28.1. Euclidean connexions 275
28.2. Covariant differentiation of tensors of arbitrary rank 284
§29. Covariant differentiation and the metric 288
29.1. Parallel transport of vector fields 288
29.2. Geodesics 290
29.3. Connexions compatible with the metric 291
29.4. Connexions compatible with a complex structure (Hermitian metric) 295
29.5. Exercises 297
§30. The curvature tensor 299
30.1. The general curvature tensor 299
30.2. The symmetries of the curvature tensor. The curvature tensor defined by the metric 304
30.3. Examples: The curvature tensor in spaces of dimensions 2 and 3; the curvature tensor of transformation groups 306
30.4. The Peterson–Codazzi equations. Surfaces of constant negative curvature, and the "sine–Gordon" equation 311
30.5. Exercises 314

CHAPTER 5

The Elements of the Calculus of Variations 317
§31. One-dimensional variational problems 317
31.1. The Euler–Lagrange equations 317
31.2. Basic examples of functionals 321
§32. Conservation laws 324
32.1. Groups of transformations preserving a given variational problem 324
32.2. Examples. Applications of the conservation laws 326
§33. Hamiltonian formalism 337
33.1. Legendre's transformation 337
33.2. Moving co-ordinate frames 340
33.3. The principles of Maupertuis and Fermat 345
33.4. Exercises 348
§34. The geometrical theory of phase space 348
34.1. Gradient systems 348
34.2. The Poisson bracket 352
34.3. Canonical transformations 358
34.4. Exercises 362
§35. Lagrange surfaces 362
35.1. Bundles of trajectories and the Hamilton–Jacobi equation 362
35.2. Hamiltonians which are first-order homogeneous with respect to the momentum 367

§36. The second variation for the equation of the geodesics 371
36.1. The formula for the second variation 371
36.2. Conjugate points and the minimality condition 375

CHAPTER 6
The Calculus of Variations in Several Dimensions. Fields and Their Geometric Invariants 379
§37. The simplest higher-dimensional variational problems 379
37.1. The Euler–Lagrange equations 379
37.2. The energy-momentum tensor 383
37.3. The equations of an electromagnetic field 388
37.4. The equations of a gravitational field 394
37.5. Soap films 401
37.6. Equilibrium equation for a thin plate 407
37.7. Exercises 412
§38. Examples of Lagrangians 413
§39. The simplest concepts of the general theory of relativity 416
§40. The spinor representations of the groups $SO(3)$ and $O(3, 1)$. Dirac's equation and its properties 431
40.1. Automorphisms of matrix algebras 431
40.2. The spinor representation of the group $SO(3)$ 433
40.3. The spinor representation of the Lorentz group 435
40.4. Dirac's equation 439
40.5. Dirac's equation in an electromagnetic field. The operation of charge conjugation 441
§41. Covariant differentiation of fields with arbitrary symmetry 443
41.1. Gauge transformations. Gauge-invariant Lagrangians 443
41.2. The curvature form 447
41.3. Basic examples 448
§42. Examples of gauge-invariant functionals. Maxwell's equations and the Yang–Mills equation. Functionals with identically zero variational derivative (characteristic classes) 453

Bibliography 459

Index 463